The Economics of Automated Market Making

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Joint work with
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Introduction

This Talk

Central topic today: automated market makers (AMMs)
A new mechanism for electronic trading
(vs. limit order book, dark pool, batch auction, etc.)

Only tangentially relevant: cryptocurrencies, blockchain

Trading via an Order Book

Problem: Enable exchange of assets (e.g., ETH for USD and vice versa)

Traditional Solution: Central limit order book (e.g., NASDAQ, CME, Coinbase, etc.)

- Accept offers to buy/sell prescribed quantities at prescribed prices
- Match pairs of mutually acceptable orders
  All remaining buy prices $<$ all remaining sell prices

Issues:

1. Costly to store/compute "on-chain"
   Very high update rates
2. Requires active participation of market makers
   Illiquidity for "long-tail" assets

1	Type	Price (in USD/ETH)	Quantity (in ETH)
2	sell	1225	30
3	sell	1223	5
4	sell	1220	100
5	sell	1215	8
6	sell	1205	12
7	buy	1195	25
8	buy	1185	75
9	buy	1182	33
10	buy	1180	42

Automated Market Makers

Key Idea: [Buterin, Köppelman, Lu 2016; ...]

- "Liquidity providers" (LPs) supply pools of USD + ETH
- Market always willing to accept buy/sell orders at quoted price
- Automated quoting mechanism: price set by quantity of assets of each type
  Inspired by use in prediction markets [e.g., Pennock, Sami 2007]
  "Constant function market makers" (CFMMs)
- Benefit: LPs earn trading fees (% fee)
- Minimal storage needs; Computations can be done quickly, typically via closed-form
- Primarily rely on passive liquidity providers

Economics of Liquidity Provision

Motivating questions:

- How can we measure the performance of liquidity providers in AMMs / CFMMs?
- How does performance depend on asset dynamics (e.g., volatility)? Pool characteristics (e.g., bonding curve, fee structure)? Blockchain characteristics (e.g., block rate)?
- How can we improve AMM design from the LP perspective?

Working papers:

- J. Milionis, C. C. Moallemi, T. Roughgarden, A. L. Zhang. Automated market making and loss-versus-rebalancing. Working paper. Initial version: August 2022. Revised: June 2023.
- J. Milionis, C. C. Moallemi, T. Roughgarden. Automated market making and arbitrage profits in the presence of fees. Working paper. Initial version: February 2023. Revised: May 2023.
- Available at `https://moallemi.com/ciamac` or on Arxiv

Contributions (1)

- Our main contribution is a **"Black-Scholes Formula for AMMs"**
- Like Black-Scholes, we analyze delta-hedged LP returns
- Short whatever amount of ETH your USD-ETH LP position holds, at any point in time:

$$\text{Delta-Hedged LP P\&L}_T = \underbrace{\text{FEE}_T - \text{LVR}_T}_{\text{Fees Minus LVR}}$$

- "Loss-versus-rebalancing", LVR_T ("lever"), arises from slippage: stale AMM prices are picked off by arbitrageurs ("searchers")

$$\text{LVR}_T = \frac{1}{2}\int_0^T \underbrace{|x^{*\prime}(P_t)|}_{\text{marginal liquidity}} \underbrace{\sigma_t^2 P_t^2 \, dt}_{\text{quadratic variation}} \geq 0$$

- Formula works well empirically
- Suggests improved AMM designs

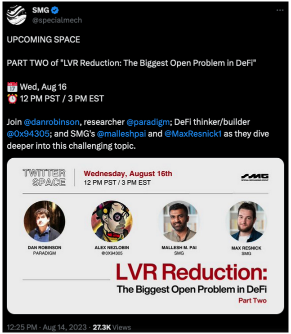
Contributions (2)

- LVR derived assuming arbitrageurs pay no fees, trade continuously
- We further derive closed-form and asymptotic expressions for arbitrage profits with trading fees and discrete, Poisson block generation:

$$\text{arb profits} \approx \text{LVR} \times \underbrace{\frac{1}{1 + \frac{\gamma}{\sigma\sqrt{\Delta t/2}}}}_{\triangleq \mathsf{P}_{\mathsf{trade}}}$$

- In the fast block regime ($\Delta t \to 0$), arb profits $= \Theta(\sqrt{\Delta t})$
- LVR $\approx$ arb profits $+$ fees paid by arbs to LPs

Loss-Versus-Rebalancing in Industry



Literature Review

- **Options Pricing / Market Microstructure**
  Black, Scholes 1973; Merton 1973; Carr, Madan 2002
  Glosten, Milgrom 1985; ...
  Budish, Cramton, Shim, 2015; ...
- **Prediction Markets**
  Winkler 1969; Savage 1971; Hanson 2002
  Chen, Pennock 2007; ...
- **AMMs for Exchanges**
  Buterin, Köppelmann, Lu 2016
  Angeris, Evans, Chitra 2020–2021
  Capponi, Jia 2021; Lehar, Parlour 2021; Park 2021; Aoyagi, Ito 2021;
  Barbon, Ranaldo 2021; O'Neill 2022; Cartea, Drissi, Monga 2022; Nezlobin 2022;
  Dewey,Newbold 2023; ...

Background: Blockchain and Decentralized Finance

What is a Blockchain?

Blockchains provide generic mechanisms for trustless consensus about distributed state machines, i.e., they are (decentralized) computers

- A general-purpose computer ("Turing complete")
- No single owner or operator ("computer-in-the-sky", a public good)
- Open access (anyone can use or deploy applications)
- Supports internal property rights (users can "own" data)

Intellectual origins of the modern blockchain:

- (Coöperative) distributed consensus
- Cryptographic primitives (e.g., hash functions, public key cryptography)
- Economics / incentives / game theory

Decentralized Computers

Bitcoin (2009)

- state transitions: payments
- consensus: account balances of a distributed ledger

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Ethereum (2015) (and most modern blockchains)

- state transitions: Turing complete! "smart contracts" = arbitrary computer programs
- consensus: shared memory of a distributed global virtual machine

But Very Slow and Expensive Computers!

Raspberry PI

- Hobbyist computer
- Unit cost: $45 (retail)
- CPU performance: 5000x

Ethereum

- Global virtual machine
- Operating cost: ~$20M/day
- CPU performance: 1x

credit: Nicholas Weaver

Rise of Decentralized Finance

Top 20 Ethereum smart contracts
Measured by resource consumption (normalized gas)

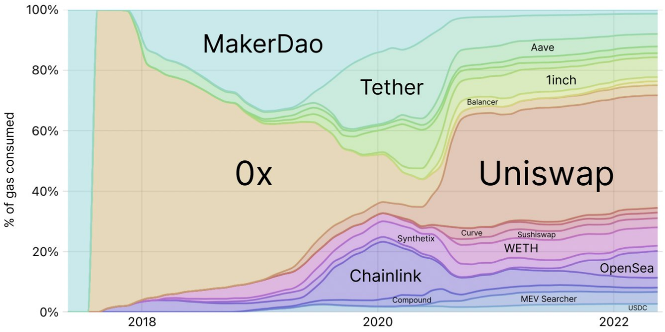
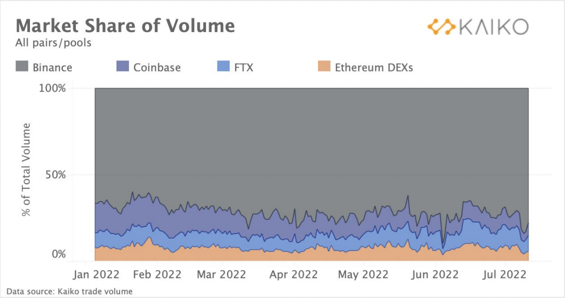


image credit: @caseykcaruso / https://gasguzzlers.wtf

- Decentralized exchanges (DEXs) (AMMs / CFMMs)
  Uniswap, Balancer, Curve, Sushiswap
- Collateralized lending
  MakerDAO, Aave, Compound
- Stablecoins
  MakerDAO, Tether, USDC
- Non-fungible tokens (NFTs)
  OpenSea

DEX Market Share in Crypto



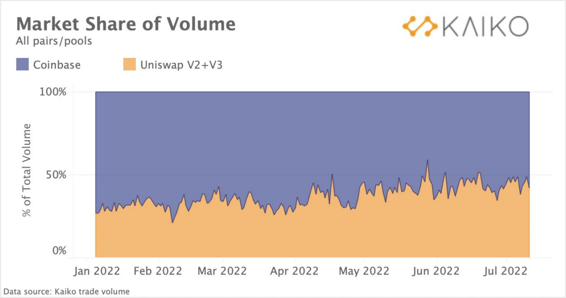


image credit: Kaiko

- Volume on Uniswap exceeds that on Coinbase
- In excess of US$1 trillion traded on Uniswap

Model

Market Model

- TL;DR: continuous time, Black-Scholes setup
- WLOG two assets: "risky" asset x (e.g., ETH), "numéraire" y (e.g., USD)
- WLOG risk-free rate $= 0$
- $P_t \triangleq$ market price of risky asset, on infinitely deep centralized exchange (CEX)
  CEX is where price discovery occurs

Market Model

- $P_t \triangleq$ market price of risky asset (on infinitely deep centralized exchange/CEX)

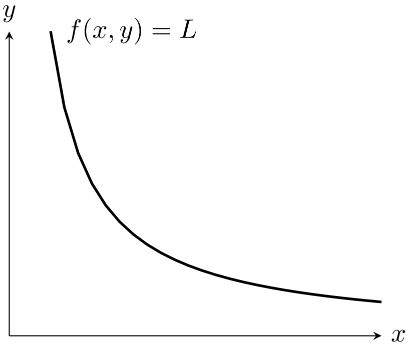
- Returns given by

$$\frac{P_{t+\Delta t} - P_t}{P_t} \approx N(\mu \Delta t, \sigma_t^2 \Delta t)$$

$$\Leftrightarrow \quad \underbrace{\frac{dP_t}{P_t}}_{\text{instantaneous return}} = \underbrace{\mu}_{\text{drift}} \times dt + \underbrace{\sigma_t}_{\text{volatility}} \times \underbrace{dB_t}_{\text{Brownian increment}}$$

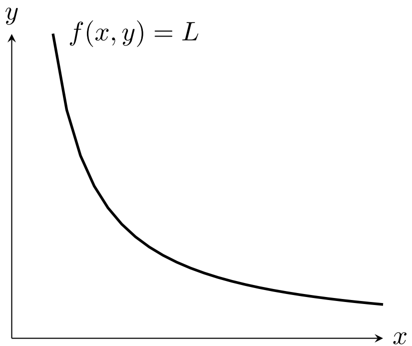
Constant Function Market Makers

- Given bonding function f



Constant Function Market Makers

- Given bonding function f
- **Example:** Constant product market maker (CPMM, Uniswap V2)
  $f(x, y) \triangleq xy$



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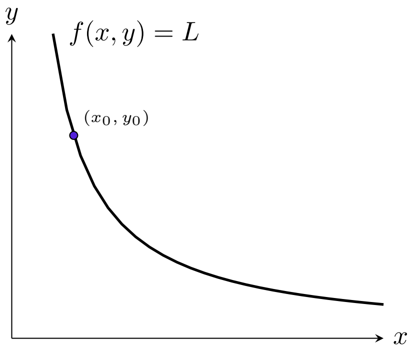

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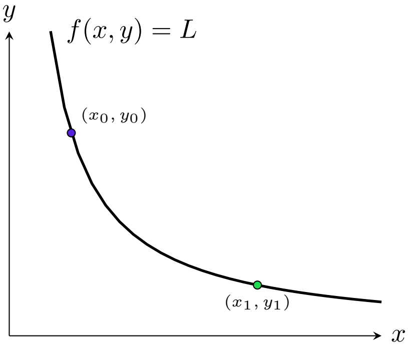

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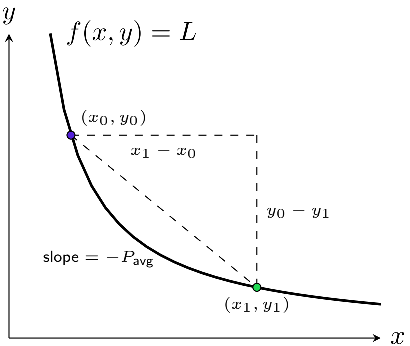
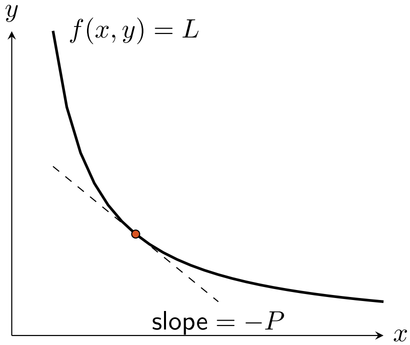
- Allow trades that maintain the invariant
  $f(x, y) = L$
- $P_{\text{avg}} = \dfrac{y_0 - y_1}{x_1 - x_0}$



Constant Function Market Makers

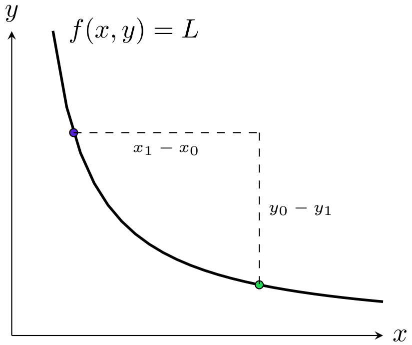
- Slope yields spot price: $P = \dfrac{\partial f / \partial y}{\partial f / \partial x}$



Constant Function Market Makers

- Fees are collected
  Proportional to traded quantity

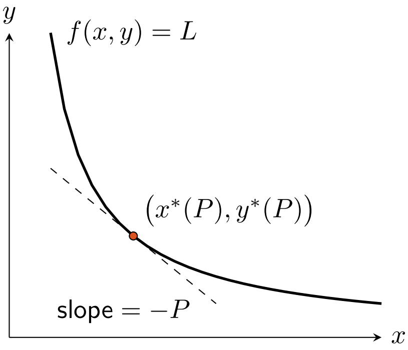
- **Example:** (Uniswap V2)
  30bp fee on contributed asset



CFMM Pool Value Function

Pool value function $V(P)$ is the monetary value of CFMM reserve holdings, when price is P, due to arbitrage:

$$V(P) \triangleq \begin{array}{ll} \underset{(x,y)\in\mathbb{R}^2_+}{\text{minimize}} & Px + y \\ \text{subject to} & f(x,y) = L \end{array}$$



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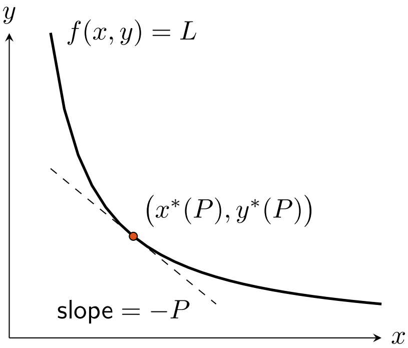

Assumption. An optimal solution $\big(x^*(P), y^*(P)\big)$ exists, and $V(\cdot)$ is twice continuously differentiable.

Example: Constant Product Market Maker

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Example. (Uniswap V2)

- Constraint set: $\left\{(x,y) \in \mathbb{R}^2_+ \ : \ f(x,y) \triangleq xy = L\right\}$
- Demand curve: $x^*(P) = L/\sqrt{P}, \quad y^*(P) = L\sqrt{P}$
- Pool value: $V(P) = Px^*(P) + y^*(P) = 2L\sqrt{P}$

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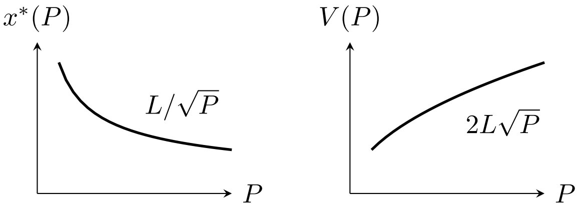
Remarks:

- $x^*(\cdot)$ is the LPs' passive demand curve for the risky asset
- $V(\cdot)$ is analogous to a "payoff function" for the pool reserves
- Setting is fully general to all passive market makers (including concentrated pools like Uniswap V3), smoothness is key requirement



Loss-Versus-Rebalancing

Market Participants

Stylized model, with two types of traders:

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- **Arbitrageurs**:
  - *Continuously* monitor the market
  - Can trade in the CFMM, or frictionlessly on infinite depth CEX
  - Hence, arb CFMM until prices equal to CEX
  - For simplicity, assume arbs *do not* pay trading fees (we will revisit!)

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  - Hence, arb CFMM until prices equal to CEX
  - For simplicity, assume arbs *do not* pay trading fees (we will revisit!)
- **Noise traders**:
  - Only trade on CFMM
  - Trade for idiosyncratic reasons (e.g., convenience of executing on-chain)
  - *Do* pay trading fees: cumulative fees FEE_t

Pool value lets us write LP P&L as:

$$\text{LP P\&L}_t = V_t - V_0 + \text{FEE}_t$$

where $V_t \triangleq V(P_t)$, $\text{FEE}_t \triangleq$ cumulative fees at t

Rebalancing Strategy

$$\text{LP P\&L}_t = \underbrace{V_t - V_0}_{\text{pool value change}} + \underbrace{\text{FEE}_t}_{\text{accumulated fees}}$$

- Decompose $V_t - V_0$ using the idea of rebalancing strategy
- Informally, the strategy makes same trades as CFMM, at external market prices

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- Decompose $V_t - V_0$ using the idea of rebalancing strategy
- Informally, the strategy makes same trades as CFMM, at external market prices
- Formally, it is is the *self-financing* trading strategy defined by:
  - Initial holdings match the pool, i.e., $(x_0, y_0) \triangleq \big(x^*(P_0), y^*(P_0)\big)$
  - Risky holdings continuously rebalanced to match the pool, i.e., $x_t \triangleq x^*(P_t)$

$$\begin{aligned} R_t &\triangleq \text{rebalancing portfolio value} \\ &\approx \underbrace{V_0}_{\text{initial value}} + \sum_{i=0}^{t/\Delta t - 1} \underbrace{x^*(P_{i\Delta t}) \times \big(P_{(i+1)\Delta t} - P_{i\Delta t}\big)}_{\text{per period P\&L}} \\ &= V_0 + \int_0^t x_s^*(P_s)\, dP_s \end{aligned}$$

Loss vs. Rebalancing

Define loss-versus-rebalancing (LVR) as:

$$\mathsf{LVR}_t \triangleq \underbrace{R_t}_{\text{rebalancing portfolio value}} - \underbrace{V_t}_{\text{pool reserve value}}$$

Intuitively: how much does V_t lose, compared to making same trades at market prices R_t?

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Theorem. (Milionis, Moallemi, Roughgarden, Zhang 2022) The LVR process is *non-negative*, *non-decreasing*, and *predictable*, and satisfies

$$\text{LVR}_t = \frac{1}{2}\int_0^t \underbrace{|x^{*\prime}(P_s)|}_{\text{marginal liquidity}} \underbrace{\sigma_s^2 P_s^2\, ds}_{\text{quadratic variation}} \geq 0$$

Note: LVR is different than “impermanent loss”!

Intuition: Slippage

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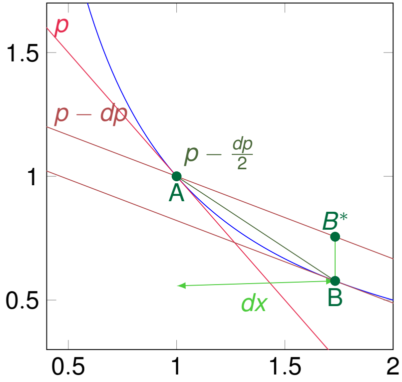
- Suppose external prices changes from p to $p - dp$



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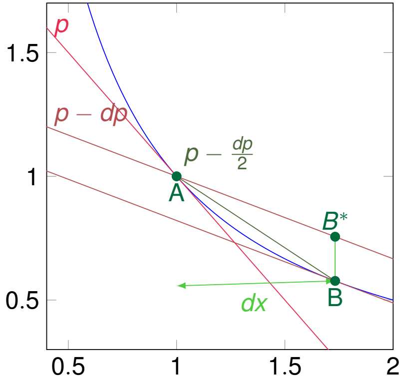

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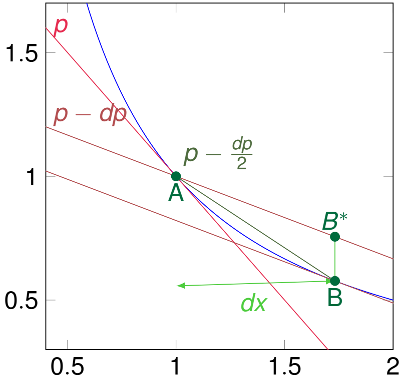

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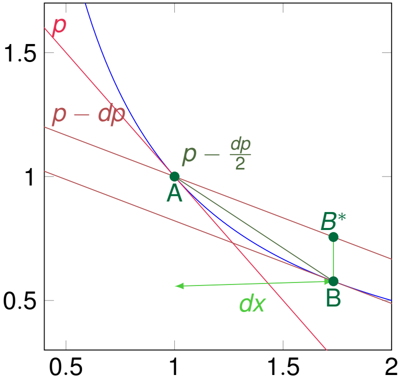
- Suppose external prices changes from p to $p - dp$
- AMM buys quantity dx
- $p_{\text{AMM}} = p - \frac{1}{2}dp$
- AMM loss/arb profit is

$$\underbrace{dx(p - \tfrac{1}{2}dp)}_{\text{AMM price}} - \underbrace{dx(p - dp)}_{\text{external price}}$$

$$= \frac{dx\, dp}{2} = \frac{1}{2}\left|\frac{dx}{dp}\right|(dp)^2 = \frac{1}{2} \times |x^{*\prime}(p)| \times \sigma^2 p^2 \, dt$$

since $(dp)^2 = \sigma^2 p^2 \, dt$ is the quadratic variation



LP Return Decomposition

Adding in fees,

$$\text{LP P\&L}_t = \underbrace{\text{FEE}_t}_{\text{accumulated fees}} + \underbrace{V_t - V_0}_{\substack{\text{change in pool}\\ \text{reserve value}}} = \underbrace{\int_0^t x^*(P_s)\, dP_s}_{\text{rebalancing P\&L}} + \underbrace{\text{FEE}_t - \text{LVR}_t}_{\text{fees minus LVR}}$$

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Like Black-Scholes, our decomposition corresponds to a tradable strategy!

- Simply delta-hedge the LP position!
- Long CFMM LP, short rebalancing portfolio, isolates $\text{FEE}_t - \text{LVR}_t$

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$$\Rightarrow \quad \text{Hedged LP P\&L}_T = \text{FEE}_T - \text{LVR}_T = \gamma \times \text{VOLUME}_T - \int_0^T \frac{\sigma_t^2 P_t^2}{2} |x^{*\prime}(P_t)|\, dt$$

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Continuously hedged LP P&L is variance-to-volume swap:

- Receive floating leg proportional to volume
- Pay floating leg of a (continuously sampled, liquidity weighted) variance swap

Example: Constant Product Market Maker

Example. (Uniswap V2)

- Constraint set: $\left\{(x, y) \in \mathbb{R}_+^2 \ : \ f(x,y) \triangleq xy = L\right\}$
- Pool value: $V(P) = 2L\sqrt{P}$ Demand curve: $x^*(P) = L/\sqrt{P}$
- Instantaneous LVR: $\dfrac{\sigma^2 P^2}{2}|x^{*\prime}(P)| = \dfrac{L\sigma^2}{4}\sqrt{P} = \dfrac{\sigma^2}{8}V(P)$
- Constant LVR per dollar of pool reserves
  (True of weighted geometric mean bonding functions, e.g., Balancer)

Example: Uniswap V2 WETH-USDC

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$$\text{24hr LVR cost (\%)} = \sigma^2/8 = 3.125 \text{ (bp)}, \quad \text{24hr LVR cost (\$)} = \$39\text{K}$$

Example: Uniswap V2 WETH-USDC

Naïve "yield" calculation:

$$\frac{\$66\text{K fees (daily)}}{\$125\text{M}} \approx 19\% \text{ (annual)}$$

Our "yield" calculation:

$$\frac{\$66\text{K} - \$39\text{K}}{\$125\text{M}} \approx 8\% \text{ (annual)}$$

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Empirical Test

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$$\underbrace{\text{LP P\&L}_t - \int_0^t x^*(P_s)\, dP_s}_{\text{Delta-Hedged LP P\&L}} = \underbrace{\text{FEE}_t - \int_0^t \frac{\sigma_s^2 P_s^2}{2} |x^{*\prime}(P_s)|\, ds}_{\text{Fees Minus LVR}}$$

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- **LHS:** Return of delta-hedged LP position (model-free!)
  - LP P&L$_t$: Directly measure pool value change $y_t + P_t x_t$, accounting for mints/burns
  - $\int_0^t x^*(P_s)\, dP_s$: Approximate by delta-hedging AMM at different discrete time horizons

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  - FEE$_t$: Trade volume times fee rate, directly measured
  - $\frac{\sigma_t^2 P_t^2}{2} |x^{*\prime}(P_t)| = \sigma_t^2 / 8 \times$ pool value for constant product MM
    Use same day 60 minute realized volatility for σ_t

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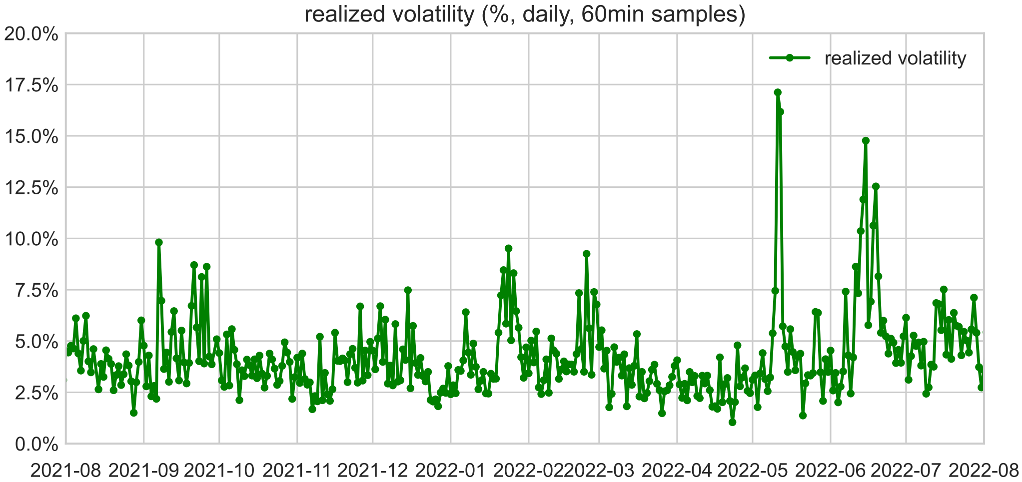
Questions:

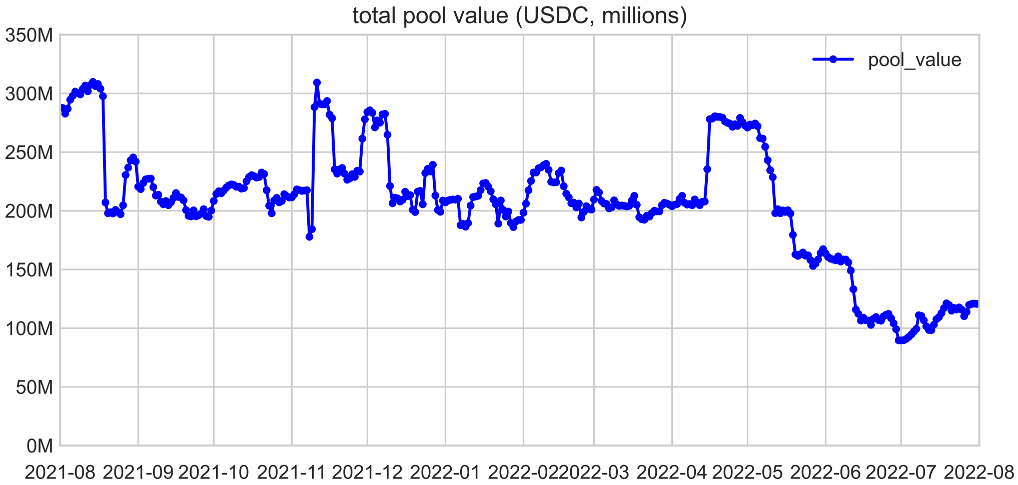
- Is our analytic formula accurate?
- Is LPing attractive?

Volatility



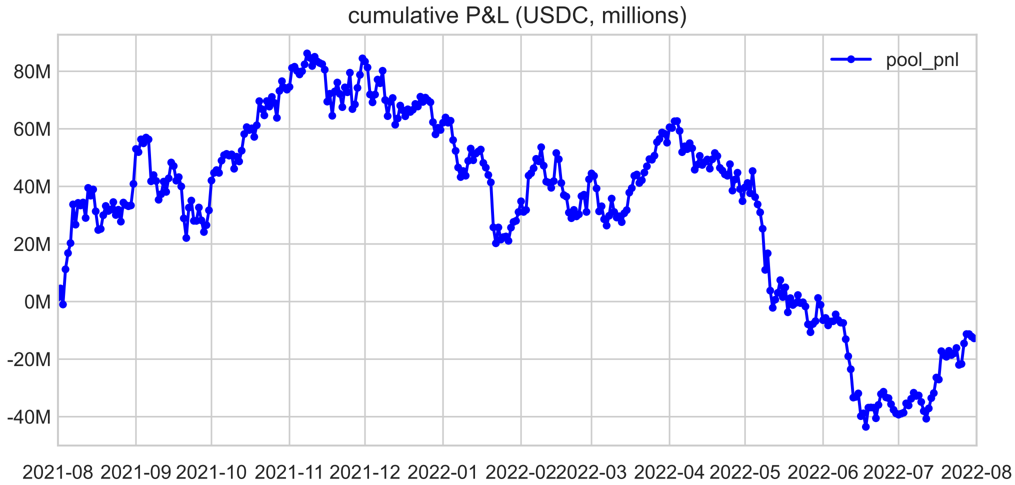
Data set: Binance ETH-USDC prices

Pool Value



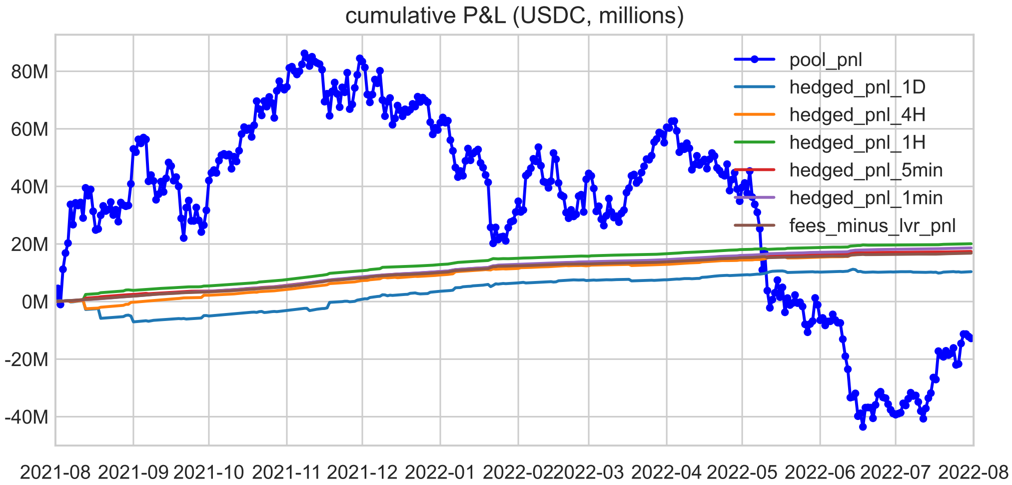
Data set: Uniswap V2 WETH-USDC pool (from Ethereum blockchain), Binance ETH-USDC prices

LP P&L



cumulative P&L (USDC, millions)
pool_pnl
80M
60M
40M
20M
0M
-20M
-40M
2021-08
2021-09
2021-10
2021-11
2021-12
2022-01
2022-02
2022-03
2022-04
2022-05
2022-06
2022-07
2022-08

Hedged P&L and LVR



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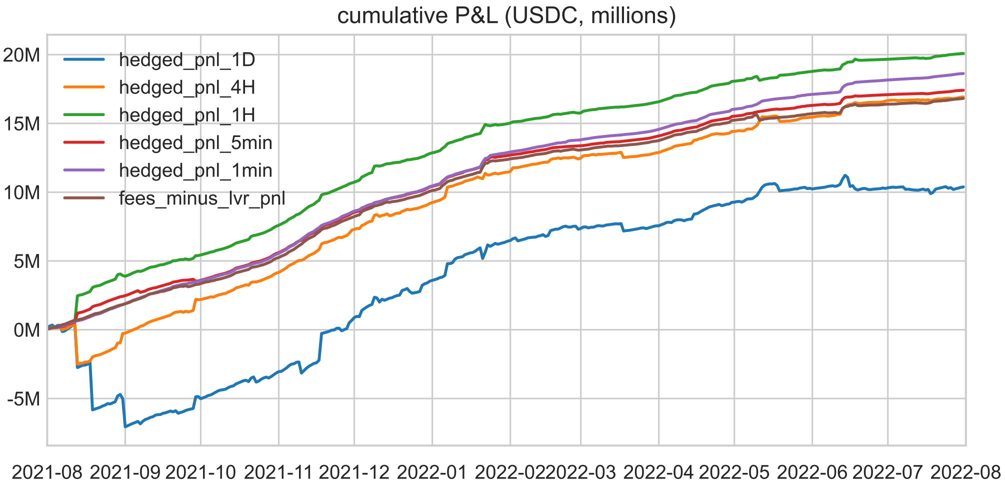

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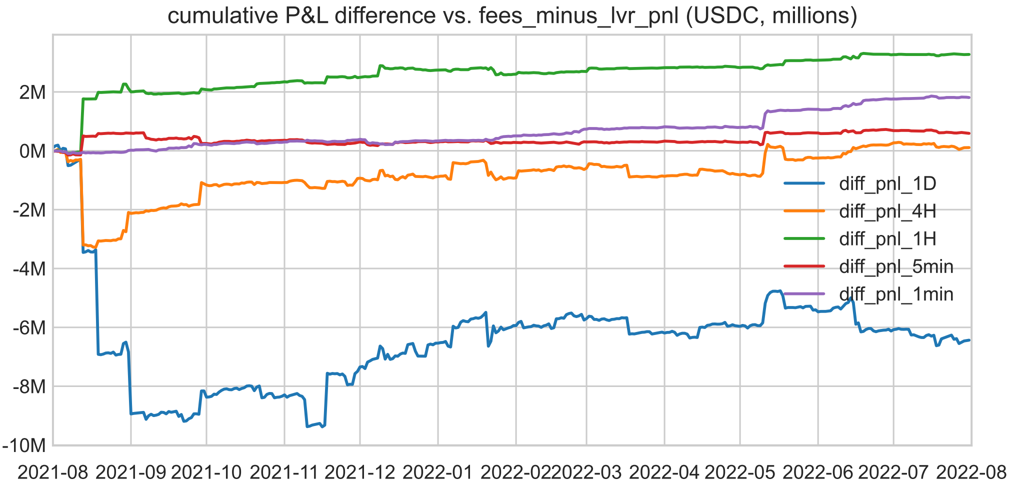
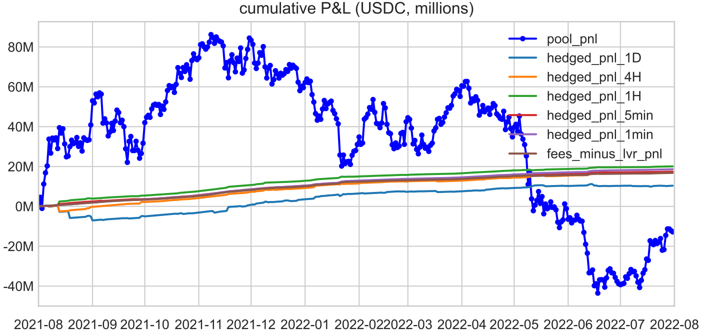


Returns



	Return	Sharpe (annualized)
Pool P&L	-6.2%	-0.2
Hedged P&L (daily)	5.0%	1.8
Hedged P&L (4 hour)	8.2%	5.5
Hedged P&L (1 hour)	9.7%	10.8
Hedged P&L (5 min)	8.4%	18.2
Hedged P&L (1 min)	9.0%	23.3
Fees-LVR	8.2%	17.0

Data set: Uniswap V2 WETH-USDC pool (from Ethereum blockchain), Binance ETH-USDC prices

What If Arbitrageurs Pay Fees?

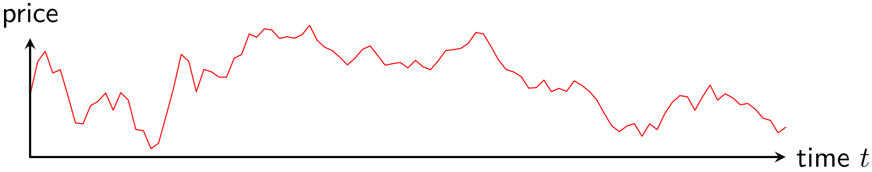
LVR **and Arbitrage Profits**

$$\text{LVR} = \text{Arbitrage Profits}$$

under the assumptions that:

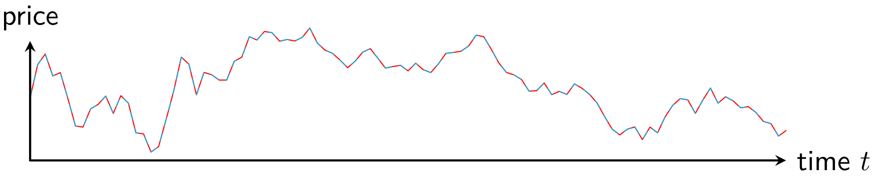
- arbitrageurs able to trade continuously
  $\Rightarrow$ in reality: can only trade at discrete instances of block generation
- arbitrageurs do not pay fees
  $\Rightarrow$ in reality: AMMs have trading fees

Impact of Arbitrageur Fees



Red = external market price

Impact of Arbitrageur Fees



Red = external market price
Blue = AMM pool price

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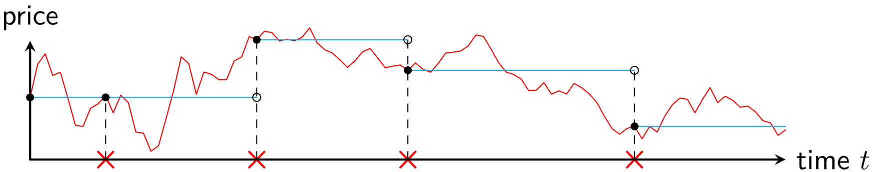

Red = external market price
Blue = AMM pool price
X = block generation times

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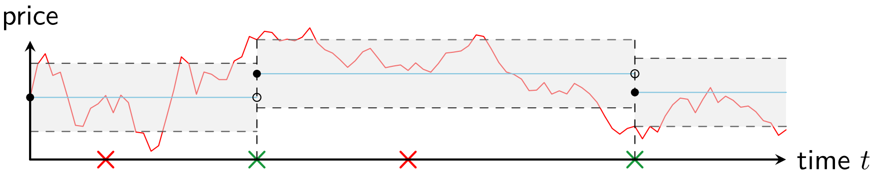

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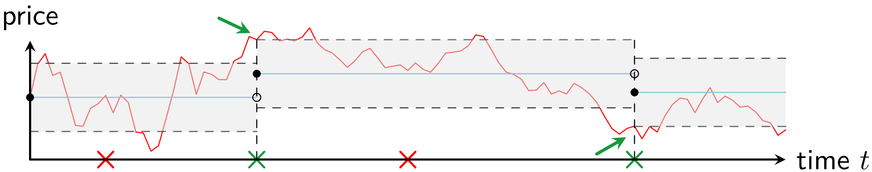

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Model (NEW)

Additional characteristics:

- Block arrival times: Poisson process with mean Δt
- Uniform proportional fees: γ fraction (e.g., 30 bp)

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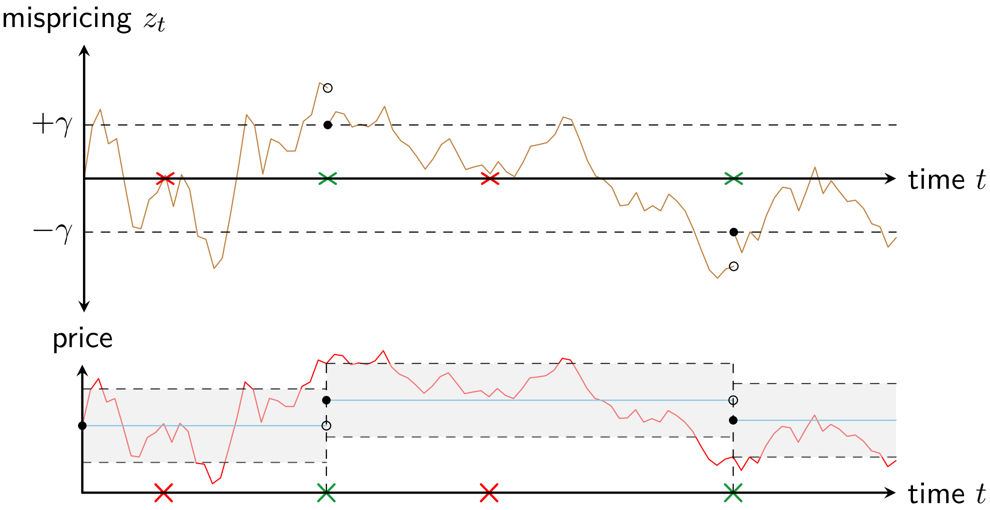
- Block arrival times: Poisson process with mean Δt
- Uniform proportional fees: γ fraction (e.g., 30 bp)

- $P_t =$ external market (CEX) price
- $\tilde{P}_t =$ implied AMM pool price
- $z_t \triangleq \log(P_t/\tilde{P}_t)$: log mispricing between pool and external market

Evolution of the Mispricing Process



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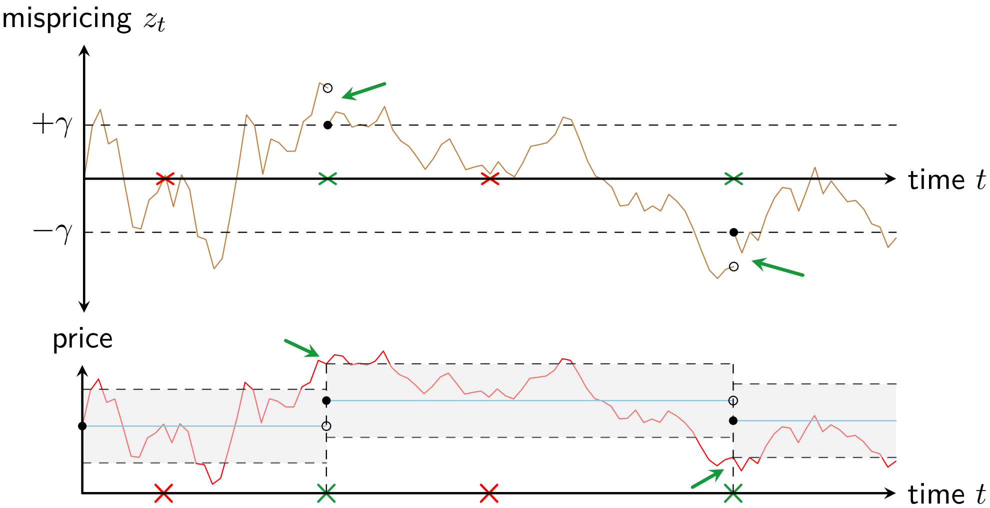

Evolution of the Mispricing Process (2)

- When a block arrives, the arb trades if $z_t \notin [-\gamma, \gamma]$ and pushes mispricing back to that boundary
- Otherwise,

$$dz_t = d \log P_t/\tilde{P}_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t$$

- z_t is a jump diffusion process
- (WLOG) Assumption (symmetry): $\mu = \frac{\sigma^2}{2}$

Fees and Discrete Block Generation

- Assume block generation $\sim$ Poisson(Δt^{-1}), $\Delta t \triangleq$ mean interblock time

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$$z_t = \begin{cases} +\gamma & \text{if } z_{t-} \geq +\gamma \\ z_{t-} & \text{if } z_{t-} \in [-\gamma, \gamma] \\ -\gamma & \text{if } z_{t-} \leq -\gamma \end{cases}$$

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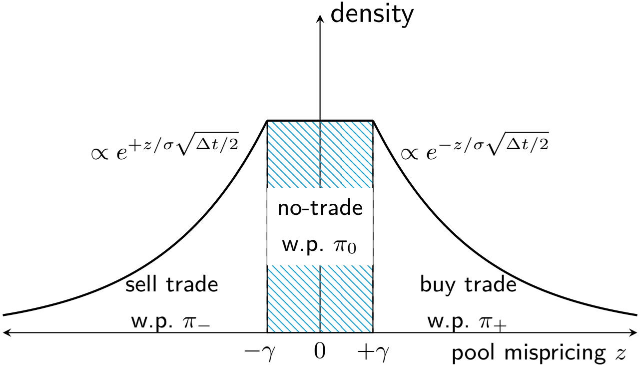
Stationary Distribution

Lemma. (Milionis, Moallemi, Roughgarden 2023) The mispricing process is ergodic, and under the symmetry assumption, the unique stationary distribution is given by:



$$\mathsf{P}_{\mathsf{trade}} \triangleq \pi_+ + \pi_- = \frac{1}{1 + \frac{\gamma}{\sigma\sqrt{\Delta t/2}}} \approx \frac{\sigma\sqrt{\Delta t/2}}{\gamma}$$

Probability of Trade

$$\mathsf{P}_{\mathsf{trade}} = \frac{1}{1 + \frac{\gamma}{\sigma\sqrt{\Delta t/2}}} = \text{fraction of blocks with an arb trade}$$

With $\sigma = 5\%$ (daily),

$\Delta t \setminus \gamma$	1 bp	5 bp	10 bp	30 bp	100 bp
10 min	96.7%	85.5%	74.7%	49.6%	22.8%
2 min	92.9%	72.5%	56.9%	30.5%	11.6%
12 sec	**80.7%**	**45.6%**	**29.5%**	**12.3%**	**4.0%**
2 sec	63.0%	25.4%	14.5%	5.4%	1.7%
50 msec	21.2%	5.1%	2.6%	0.9%	0.3%

Arbitrage Profits

- $\mathsf{ARB}_T \triangleq$ cumulative arbitrage profits over $[0, T]$
- $\overline{\mathsf{ARB}} \triangleq \lim_{T \to 0} \frac{\mathsf{E}[\mathsf{ARB}_T]}{T} =$ instantaneous intensity of arbitrage profits

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Theorem. (Milionis, Moallemi, Roughgarden 2023) Under suitable technical assumptions, in the fast block regime, as $\Delta t \to 0$,

$$\overline{\mathsf{ARB}} = \underbrace{\frac{\sigma^2 P}{2} \times \frac{y^{*\prime}\left(Pe^{-\gamma}\right) + y^{*\prime}\left(Pe^{+\gamma}\right)}{2}}_{= \overline{\mathsf{LVR}} + o(\gamma) \text{ for } \gamma \text{ small}} \times \mathsf{P}_{\mathsf{trade}} + o\left(\sqrt{\Delta t}\right)$$

$$\approx \overline{\mathsf{LVR}} \times \mathsf{P}_{\mathsf{trade}}$$

Arbitrage Profits

$$\text{intensity of arb profits } \overline{\mathsf{ARB}} \approx \overline{\mathsf{LVR}} \times \underbrace{\frac{1}{1 + \frac{\gamma}{\sigma\sqrt{\Delta t/2}}}}_{\triangleq \mathsf{P}_{\mathsf{trade}}}$$

- Equivalent to a rescaling of time by $\mathsf{P}_{\mathsf{trade}}$
- For small Δt (i.e., fast blocks), if fee rate $\gamma > 0$, $\overline{\mathsf{ARB}} = \Theta(\sqrt{\Delta t})$
- Corollary: Faster blocks $\Rightarrow$ less LP losses due to arbitrage
- Example: if $\Delta t = 12$ seconds $\rightarrow$ 3 seconds, arbitrage profits reduced by 50%
- Intuition: faster blocks create more intense competition between arbs
- Discontinuity: if fee rate $\gamma = 0$, $\overline{\mathsf{ARB}} \approx \overline{\mathsf{LVR}} = \Theta(1)$

Fees Paid by Arbs

- $\mathsf{FEE}_T^{\mathsf{ARB}} \triangleq$ cumulative fees paid by arbitrageurs over $[0, T]$

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$$\approx \overline{\mathsf{LVR}} \times (1 - \mathsf{P}_{\mathsf{trade}})$$

Fees and Discrete Block Generation

$$\text{intensity of arb profits } \overline{\mathsf{ARB}} \approx \overline{\mathsf{LVR}} \times \mathsf{P}_{\mathsf{trade}}$$

$$\text{intensity of fees paid by arbs } \overline{\mathsf{FEE}}^{\mathsf{ARB}} \approx \overline{\mathsf{LVR}} \times (1 - \mathsf{P}_{\mathsf{trade}})$$

$$\overline{\mathsf{ARB}} + \overline{\mathsf{FEE}}^{\mathsf{ARB}} \approx \overline{\mathsf{LVR}}$$

- LVR is “conserved”, fees serve to divide LVR between profits earned by arbitrageurs and fees paid by arbitrageurs to LPs

- Our techniques can be applied to other fee structures!

Implications for AMM Design

Mitigating Arbitrage Profits

Arbitrage profits are a zero-sum cost paid to intermediaries, reducing arb profits will increase gains from trade and thus social welfare

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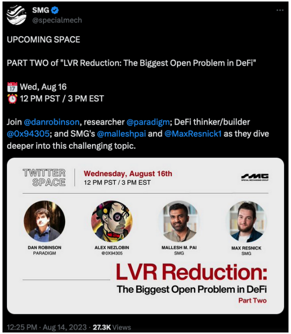


Mitigating Arbitrage Profits

Faster Blockchains

- Reduce losses to arbs potentially at the cost of less decentralization

Dynamic Fees

- Adjust fees based on market conditions (e.g., volatility/variance/LVR)
- More complex fee rules (e.g., non-proportional fees)

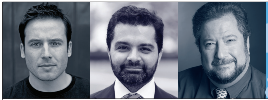
Oracle AMMs

- Incorporate external market prices into AMM quoted price

Auctions / Monetize LVR

- Auction the right to arb the pool (e.g., first trade in every block) in exchange for compensation to LPs

There is Room for Innovation in Exchange Design!



Richard Dewey, Ciamac Moallemi and Aaron Brown

Free Exchange is Not Free

The rise of crypto markets and smart contracts has fueled innovation in exchange mechanisms. This article explores core market design principles and their tradeoffs.

Instead, the exchange waits for either fixed time intervals, or until some liquidity threshold is met (such as at least US$1 million of executable orders). FBAs were previously used in equity markets such as Taiwan and have been advocated by academics as a means of mitigating the so-called "HFT tax".

It's natural to think that all three types of exchanges could co-exist, competing for trades. However, this fragments liquidity and may prevent each exchange from getting the necessary diversity of trader types. So, there are both economic forces and social benefits to concentrating transactions on one exchange type for each asset class, leaving the other exchange types to pick up niche business.

- R. Dewey, C. C. Moallemi, A. Brown. Free exchange is not free. *Wilmott Magazine*, September 2023.

End

Loss Versus Rebalancing: Proof

$$V(P) \triangleq \begin{array}{ll} \underset{(x,y)\in\mathbb{R}^2_+}{\text{minimize}} & Px + y \\ \text{subject to} & f(x,y) = L \end{array}$$

Lemma.

1. $V'(P) = x^*(P) \geq 0$
2. $V''(P) = x^{*\prime}(P) \leq 0$

Proof.

1. "Envelope Theorem": chain rule + first-order-conditions + implicit function theorem
$$V'(P) = \frac{d}{dP} \underbrace{\{Px^*(P) + y^*(P)\}}_{V(P)} = x^*(P)$$
2. Pointwise minimum of linear functions is concave

□

Loss Versus Rebalancing: Proof

$$V(P) \triangleq \underset{(x,y)\in\mathbb{R}^2_+}{\text{minimize}} \;\; Px + y, \quad \text{subject to } f(x,y) = L$$

$$V'(P) = x^*(P) \geq 0, \quad V''(P) = x^{*\prime}(P) \leq 0$$

- By Itô's lemma:

$$dV_t = V'(P_t)\,dP_t + \tfrac{1}{2}V''(P_t)\,(dP_t)^2 = x^*(P_t)\,dP_t + \tfrac{1}{2}x^{*\prime}\,(P_t)\;\sigma_t^2 P_t^2\,dt$$

- Compare rebalancing strategy:

$$R_t = V_0 + \int_0^t x_s^*(P_s)\,dP_s, \quad dR_t = x^*(P_t)\,dP_t$$

- Difference is:

$$dR_t - dV_t = -\tfrac{1}{2}x^{*\prime}\,(P_t)\;\sigma_t^2 P_t^2\,dt$$

- Intuition: LVR arises from Itô's lemma and concavity of $V(P)$, which depends on marginal liquidity $x^{*\prime}\,(P)$

Doob-Meyer Interpretation

Because of concavity/Jensen's Inequality,

$$\mathsf{E}^{\mathbb{Q}}\left[V_t|\mathcal{F}_s\right] = \mathsf{E}^{\mathbb{Q}}\left[V(P_t)|\mathcal{F}_s\right] \leq V\left(\mathsf{E}^{\mathbb{Q}}\left[P_t|\mathcal{F}_s\right]\right) = V(P_s) = V_s$$

$\Rightarrow$ pool value is a $\mathbb{Q}$-supermartingale

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$\Rightarrow$ pool value is a $\mathbb{Q}$-supermartingale

The Doob-Meyer Decomposition yields a *unique* decomposition of a supermartingale $V_t = M_t - A_t$ where:

- M_t is a martingale
- A_t is a predictable, increasing process with $A_0 = 0$ (the "compensator")

Here,

$$M_t = R_t, \qquad A_t = \mathsf{LVR}_t$$

Option Pricing Interpretation

$$\text{LP P\&L}_t = \underbrace{\text{FEE}_t}_{\text{accumulated fees}} + \underbrace{V_t - V_0}_{\substack{\text{change in pool} \\ \text{reserve value}}} = \underbrace{\int_0^t x^*(P_s)\, dP_s}_{\text{rebalancing P\&L}} + \underbrace{\text{FEE}_t - \text{LVR}_t}_{\text{fees minus LVR}}$$

Suppose we hold fixed an investment in a CFMM over $[0, T]$. What is the fair value?

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Suppose we hold fixed an investment in a CFMM over $[0, T]$. What is the fair value?

$$\mathsf{E}^{\mathbb{Q}}[\text{LP P\&L}_t] = \underbrace{\mathsf{E}^{\mathbb{Q}}[\text{FEE}_t]}_{\substack{\text{value of} \\ \text{accumulated fees}}} + \underbrace{\mathsf{E}^{\mathbb{Q}}[V(P_t)]}_{\substack{\text{future value} \\ \text{of reserves}}} - \underbrace{V(P_0)}_{\substack{\text{intrinsic current value} \\ \text{of reserves}}} = \mathsf{E}^{\mathbb{Q}}[\text{FEE}_t] - \underbrace{\mathsf{E}^{\mathbb{Q}}[\text{LVR}_t]}_{\text{time value}}$$

Option Pricing Interpretation

$$\text{LP P\&L}_t = \underbrace{\text{FEE}_t}_{\text{accumulated fees}} + \underbrace{V_t - V_0}_{\substack{\text{change in pool}\\ \text{reserve value}}} = \underbrace{\int_0^t x^*(P_s)\, dP_s}_{\text{rebalancing P\&L}} + \underbrace{\text{FEE}_t - \text{LVR}_t}_{\text{fees minus LVR}}$$

Suppose we hold fixed an investment in a CFMM over $[0, T]$. What is the fair value?

$$\mathsf{E}^{\mathbb{Q}}[\text{LP P\&L}_t] = \underbrace{\mathsf{E}^{\mathbb{Q}}[\text{FEE}_t]}_{\substack{\text{value of}\\ \text{accumulated fees}}} + \underbrace{\mathsf{E}^{\mathbb{Q}}[V(P_t)]}_{\substack{\text{future value}\\ \text{of reserves}}} - \underbrace{V(P_0)}_{\substack{\text{intrinsic current value}\\ \text{of reserves}}} = \mathsf{E}^{\mathbb{Q}}[\text{FEE}_t] - \underbrace{\mathsf{E}^{\mathbb{Q}}[\text{LVR}_t]}_{\text{time value}}$$

- $\mathsf{E}^{\mathbb{Q}}[\text{LVR}_t]$ is the fair time value of the option premium associated with the liquidity demand curve $x^*(\cdot)$ / concave payoff $V(\cdot)$
- LPs pre-commit to a liquidity curve / concave payoff, LPs receive fee income instead of an option premium
- Alternative viewpoint (vs. arb profits)

Option Pricing Interpretation

$$V(P_T) - V(P_0) = R_T - R_0 - \mathsf{LVR}_T = \int_0^T x^*(P_t)\, dP_t - \int_0^T \frac{\sigma_t^2 P_t^2}{2} |x^{*\prime}(P_t)|\, dt$$

Three ways to get exposure to volatility over the period $[0, T]$ [Carr, Madan 2002]:

- Static terminal payoff: pool reserves $V(P_T) - V(P_0)$
- Dynamic trading (delta hedging): rebalancing strategy $R_T - R_0$
- Variance swap: LVR_T

Other Benchmarks / Impermanent Loss

Consider an alternative benchmark:

- Initial holdings match the pool, i.e., $(x_0^{\mathsf{HODL}}, y_0^{\mathsf{HODL}}) \triangleq (x^*(P_0), y^*(P_0))$
- Risky holdings held constant $x_t^{\mathsf{HODL}} \triangleq x^*(P_0)$
- $\mathsf{IL}_t \triangleq \underbrace{x_0^{\mathsf{HODL}} P_t + y_0^{\mathsf{HODL}}}_{\text{HODL value}} - V_t =$ "impermanent loss" or loss-versus-holding

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Then,

$$\mathsf{IL}_t = \mathsf{LVR}_t + \int_0^t \left[x_0^{\mathsf{HODL}} - x^*(P_s) \right] dP_s$$

- *Ex ante:* $\mathsf{E}^{\mathbb{Q}}[\mathsf{IL}_t] = \mathsf{E}^{\mathbb{Q}}[\mathsf{LVR}_t]$, i.e., same "market price"
- *Ex post:* IL conflates adverse selection (LVR) with market risk
- The rebalancing portfolio is the unique choice of benchmark relative to which losses are predictable and non-decreasing
("super-replicating portfolio", compensator in Doob-Meyer Decomposition)

Example: Uniswap V3

Example. (Uniswap V3 Range Order)

- Consider a single range order over $[P_a, P_b]$ with liquidity L
- Pool value, for $P \in [P_a, P_b]$:

$$V(P) = L\left(2\sqrt{P} - P/\sqrt{P_b} - \sqrt{P_a}\right) = L\sqrt{P}\left(\frac{\sqrt{P_b} - \sqrt{P}}{\sqrt{P_b}} + \frac{\sqrt{P} - \sqrt{P_a}}{\sqrt{P}}\right)$$

- Instantaneous LVR: $\ell(\sigma, P) = \frac{L\sigma^2}{4}\sqrt{P} \Rightarrow$ same as before
- Instantaneous LVR per dollar of reserves can be arbitrarily high over a narrow range

$$\lim_{|P_b - P_a| \to 0} \frac{\ell(\sigma, P)}{V(P)} = +\infty$$