Drawdown Beta and Portfolio Optimization
Part 1

Stan Uryasev and Rui Ding

Based on Chekhlov, A., Uryasev, S., and M. Zabarankin. Drawdown Measure in Portfolio Optimization. International Journal of Theoretical and Applied Finance, V. 8, # 1, 2005

OUTLINE

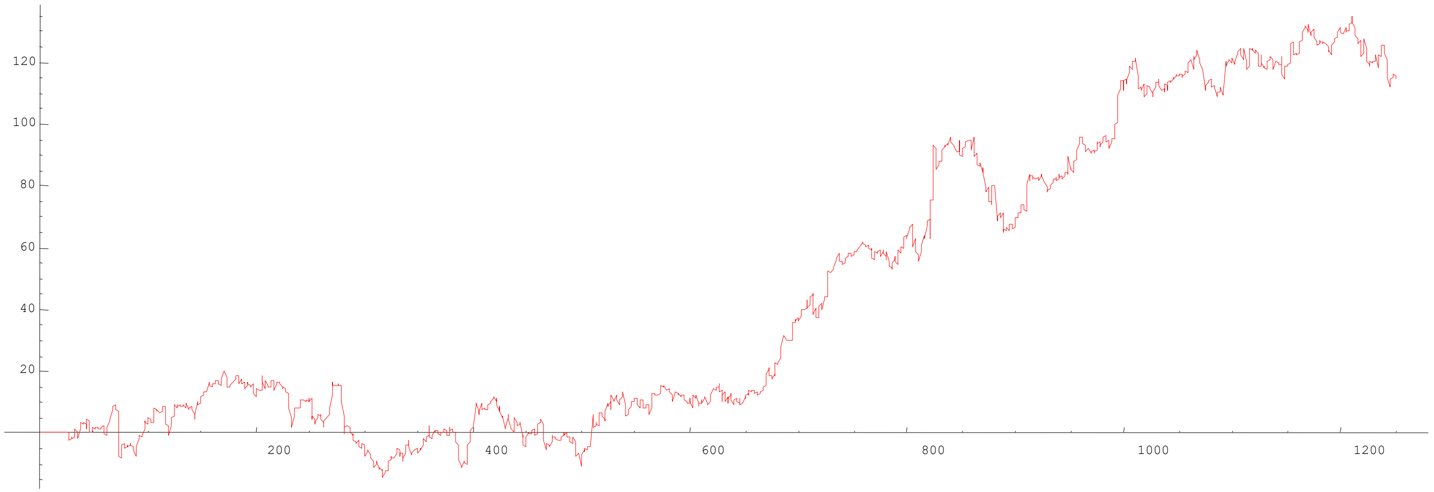
- **Motivation: drawdown is a key factor in active portfolio management**

- **Risk measures based on a drawdown of a portfolio**

- **Conditional Drawdown-at-Risk is an extension of Conditional Value-at-Risk**

- **Case study for a portfolio of futures**

MOTIVATION

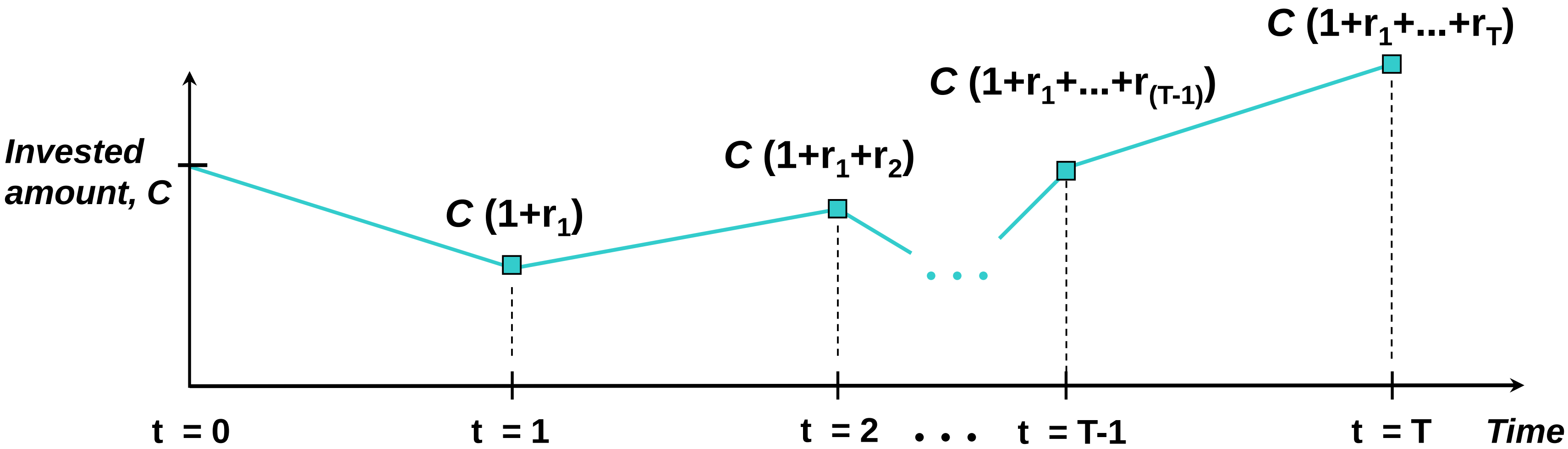


- **”Drawdown”   in active portfolio management**
  - **Highly unlikely to hold an account which was in a drawdown for 2 years**
  - **Highly unlikely to be permitted to have a 50% drawdown**
  - **Shutdown condition: 20% drawdown**
  - **Warning condition: 15% drawdown**
  - **Longest time to get out of a drawdown = 1 year**

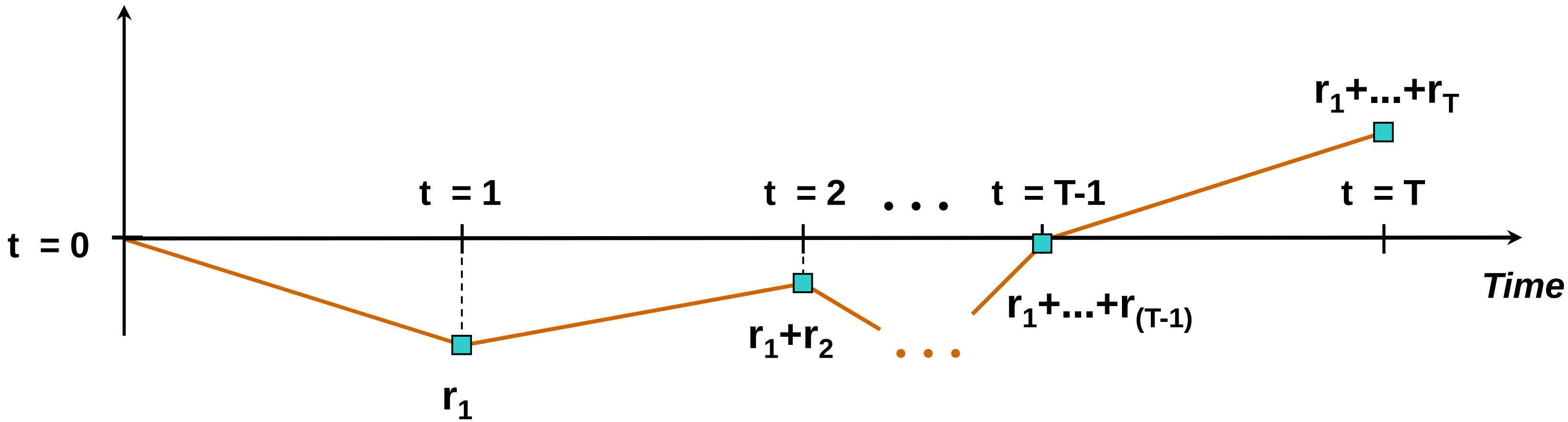
- **Classic portfolio optimization techniques do not take explicitly into account portfolio drawdown**

DEFINITION OF DRAWDOWN

Uncompounded portfolio value



Uncompounded cumulative portfolio rate of return



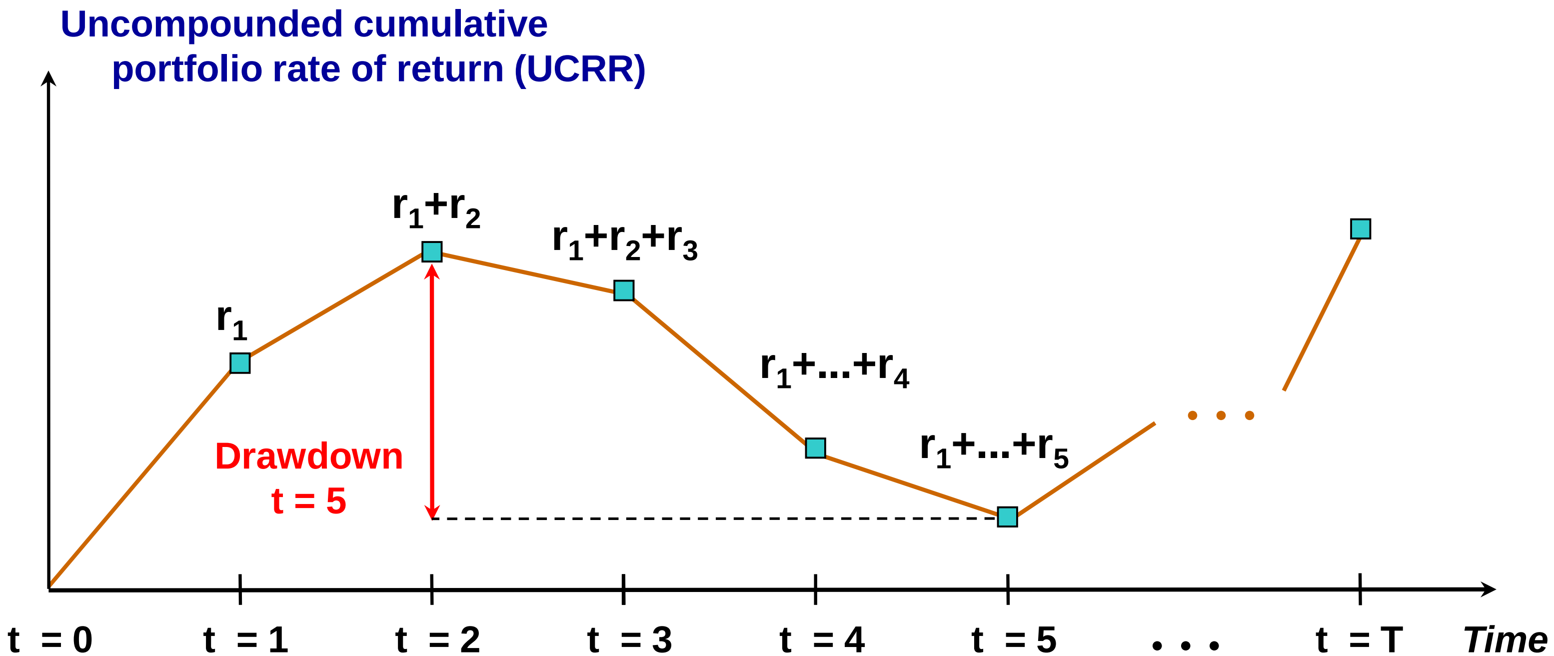
DEFINITION OF DRAWDOWN (cont'd)

Notations:

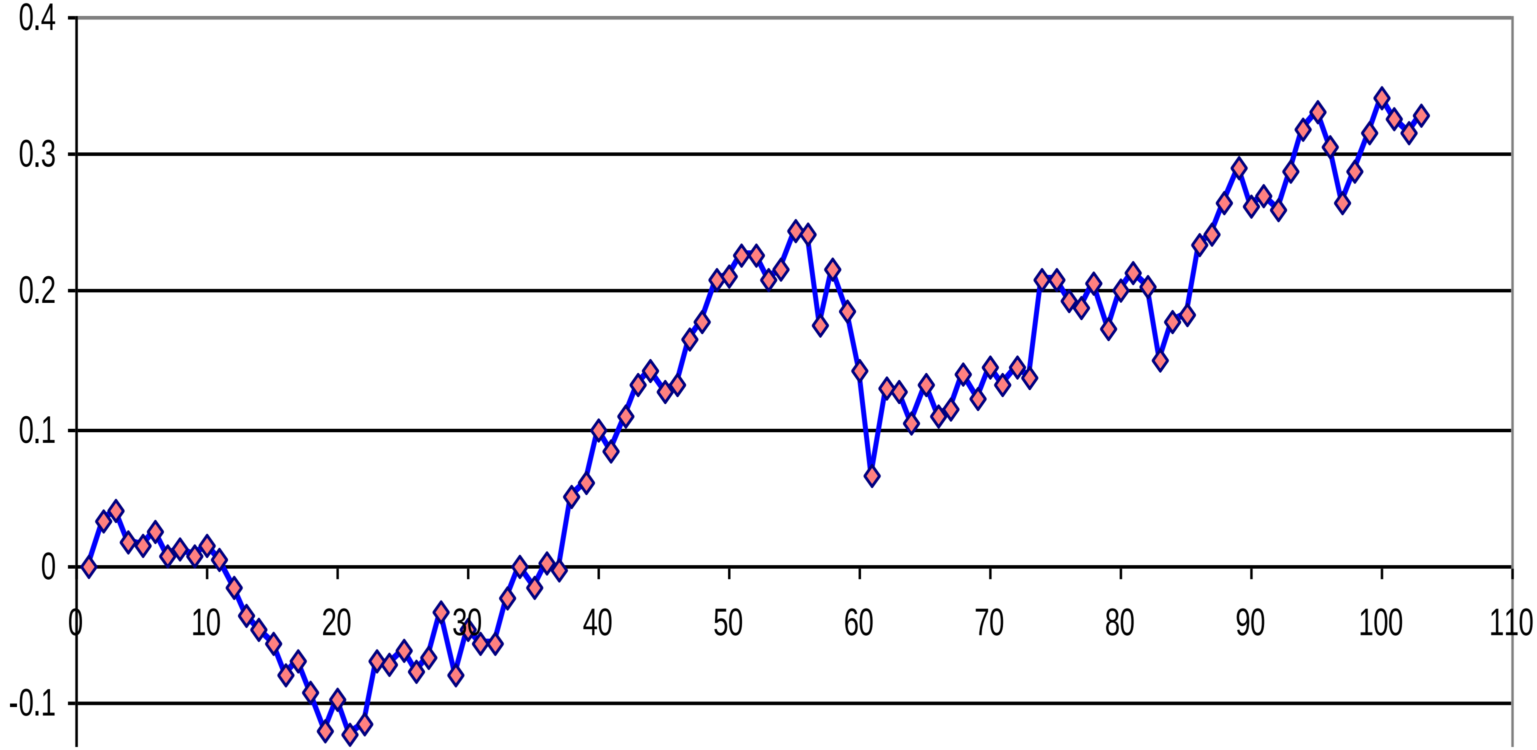
t = current moment of time,

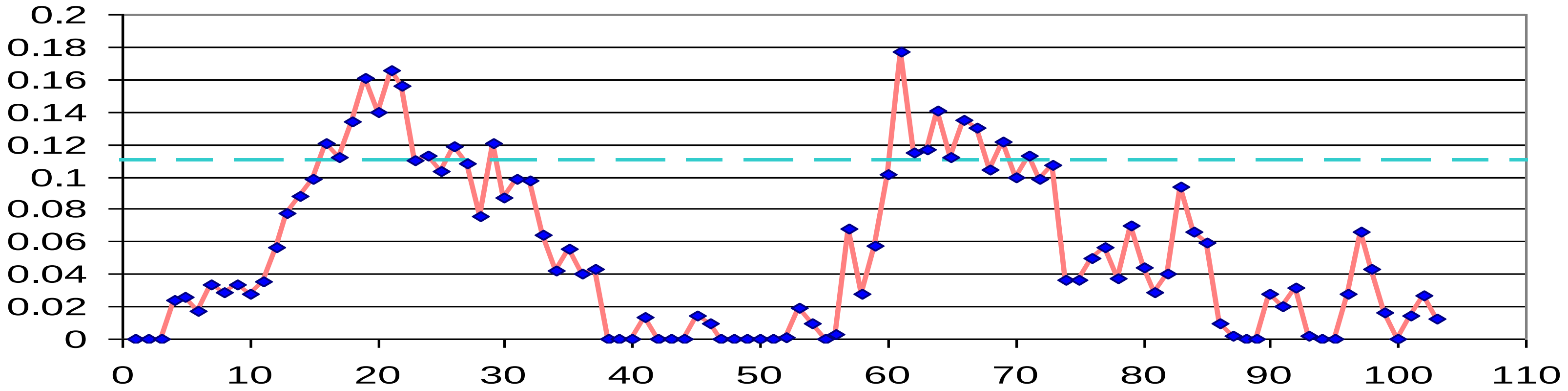
$r(t) = \sum_{\tau=0}^{t} r_\tau =$ uncompounded cumulative portfolio (random) rate of return at time t

$$DD(t) = \max_{0 \le \tau \le t} \{ r(\tau) \} - r(t) =$$ drawdown (of rate of return) at time t



Uncompounded Cumulative Portfolio Rate of Return

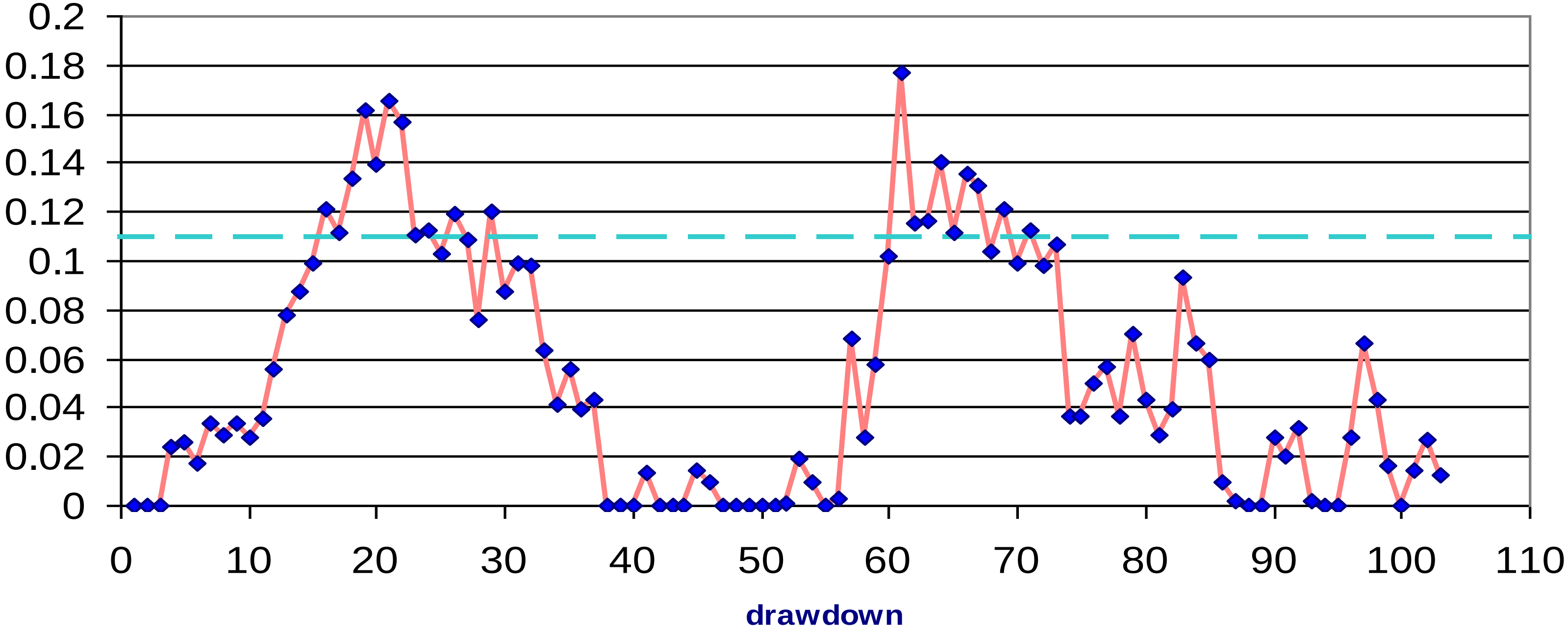




Drawdown

CDaR METHODOLOGY

- **Sample- path functions:**
  - **<u>Maximum drawdown</u> (MaxDD)**
  - **<u>Average drawdown</u> (AvDD)**
  - **<u>Conditional drawdown-at-risk</u> (CDaR)**



CDaR METHODOLOGY

t = time, T= last inivestment interval, ω = scenario (sample-path) from set Ω

$DD(t,\omega) =$ drawdown on scenario ω at time t

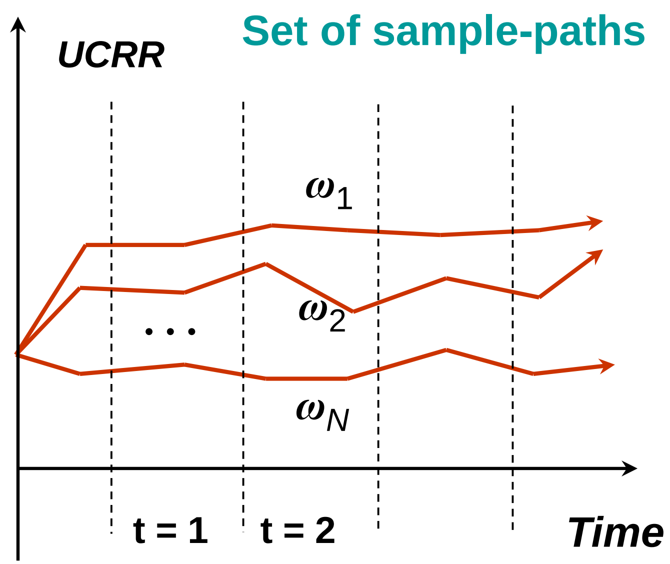
α = confidence level; (1- α)T = an integer number (this assumption will be relaxed)

Sample- path functions based on notion of drawdown:

- Maximum drawdown (MaxDD): $$\mathbf{M}(\omega) = \max_{0 \le t \le T} \{DD(t,\omega)\}$$
- Average drawdown (AvDD) : $$\mathbf{A}(\omega) = \frac{1}{T}\int_0^T DD(t,\omega)\,dt$$
- Conditional drawdown-at-risk (CDaR)

$$\Delta_\alpha(\omega) = \frac{1}{(1-\alpha)T} \int_{DD(t,\omega) \ge \varsigma(\alpha)} DD(t,\omega)\,dt$$

RISK MEASURES BASED ON NOTION OF DRAWDOWN



MaxDD

$$\mathbf{M} = \max_{t \in [0,T], \omega \in \Omega} \{DD(t,\omega)\}$$

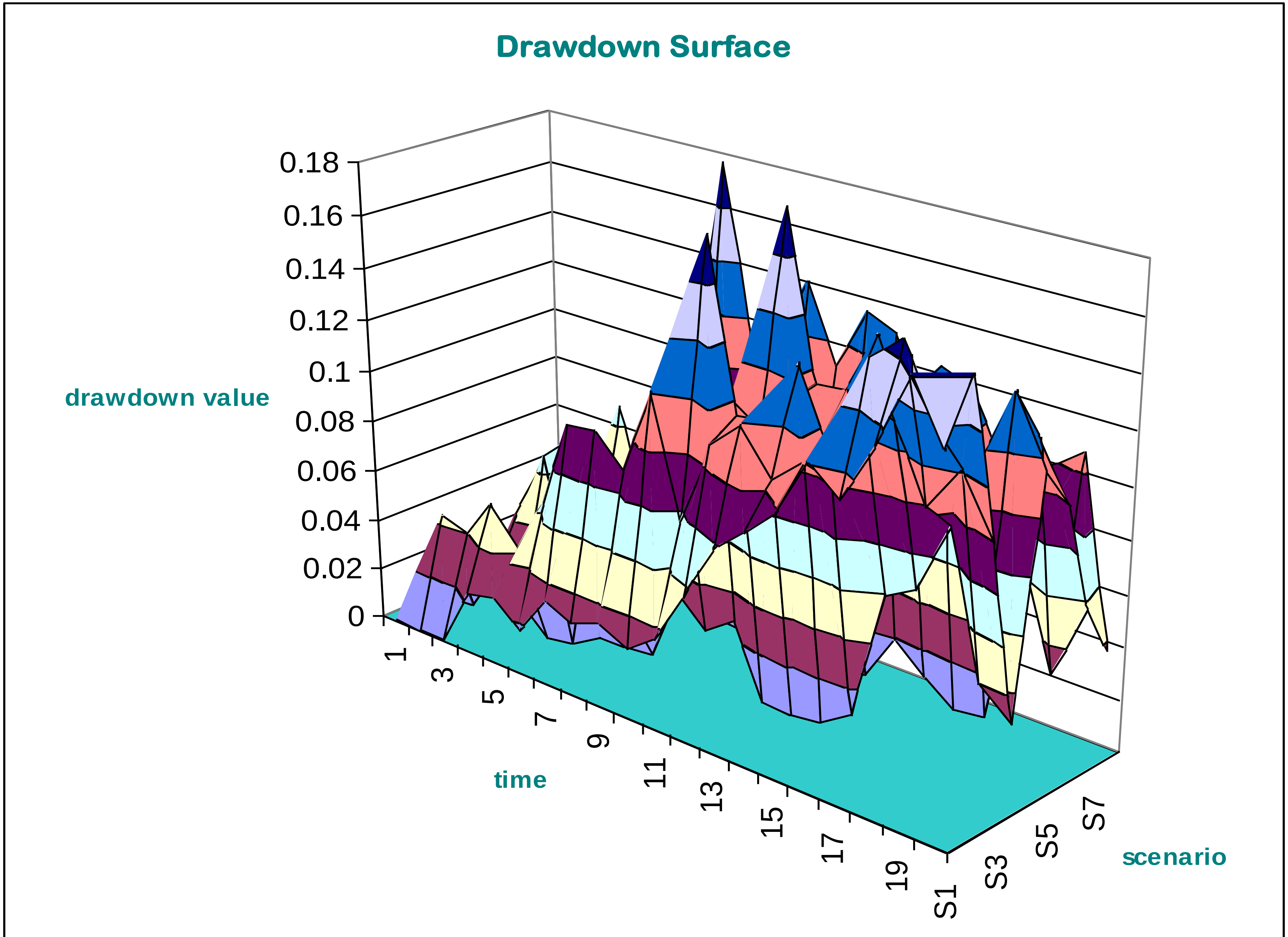
AvDD

$$\mathbf{A} = \frac{1}{T} \int_{t \in [0,T], \omega \in \Omega} DD(t,\omega)\, dF(\omega)\, dt$$

CDaR

$$\Delta_\alpha = \frac{1}{(1-\alpha)T} \int_{t \in [0,T], \omega \in \Omega : DD(t,\omega) \geq \varsigma(\alpha)} DD(t,\omega)\, dF(\omega)\, dt$$

RISK MEASURES BASED ON NOTION OF DRAWDOWN



CDaR PROPERTIES

Proposition

CDaR, MaxDD and AvDD satisfies properties of *generalized deviation measure*[1] as functionals of the random process r (*i.e.* set of sample-paths)

F denotes **CDaR, MaxDD or AvDD**

1. *Positiveness:* $F(r) \geq 0$

2. *Constant neglect:* $F(r + const) = F(r)$

3. *Homogeneity:* $F(\lambda r) = \lambda F(r), \quad \lambda > 0$

4. *Convexity:* for any r_1, r_2 and $r_\lambda = \lambda r_1 + (1-\lambda) r_2$ *with* $0 \leq \lambda \leq 1$

$$F(r_\lambda) \leq \lambda F(r_1) + (1-\lambda) F(r_2)$$

CDaR IS AN EXTENSION OF CVaR

- MaxDD and AvDD are the limiting cases of CDaR:

$$\mathrm{MaxDD} = \lim_{\alpha \to 1} \Delta_\alpha , \qquad \mathrm{AvDD} = \lim_{\alpha \to 0} \Delta_\alpha$$

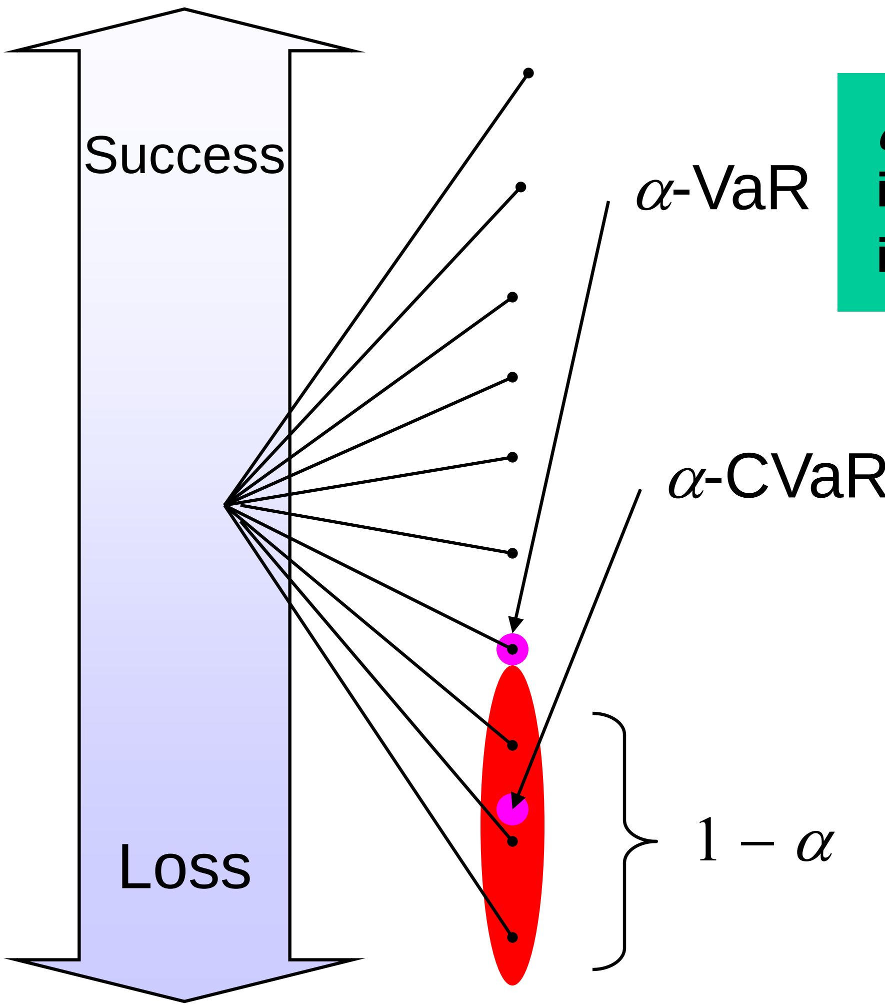
- CDaR is CVaR with the drawdown loss function.
- CDaR methodology is an extension of the CVaR approach developed in [1], [2].

[1] Rockafellar R.T. and S. Uryasev (2002): *Conditional Value-at-Risk for General Loss Distributions.* Journal of Banking and Finance, 26/7
(download : www.ise.ufl.edu/uryasev/cvar2_jbf.pdf)

[2] Rockafellar R.T. and S. Uryasev (2000): *Optimization of Conditional Value-at-Risk.* The Journal of Risk. Vol. 2, No. 3, 21-41.
(download: www.ise.ufl.edu/uryasev/cvar.pdf)

INFORMAL DEFINITION OF CVaR

- **Suppose we have a set of scenarios representing possible future gains/losses**



α-quantile of loss distribution, i.e., a loss that is exceeded in (1-α) 100% of cases

approximately, a-CVaR is the average loss in (1-α) 100% worst scenarios

RISK MANAGEMENT WITH CVaR CONSTRAINTS

- **CVaR constraints in optimization problems can be replaced by a set of linear constraints. E.g., the following <u>CVaR constraint</u>**

$$\text{CVaR} \le C$$

<u>can be replaced by linear constraints</u>

$$\zeta + \nu \sum_{\{ j=1,\dots,J \}} z_j \le C$$

$$z_j \ge f(x, y^j) - \zeta, \quad z_j \ge 0, \quad j=1,\dots J \qquad (\nu = ((1-\alpha)J)^{-1} = \text{const})$$

- **Loss distribution can be shaped using multiple CVaR constraints at different confidence levels in different times**

- **The reduction of the CVaR risk management problems to LP is a relatively simple fact following from possibility to replace CVaR by some function $F(x,\zeta)$, which is convex and piece-wise linear with respect to x and ζ . A simple explanation of CVaR optimization approach can be found in paper[1] .**

[1]Uryasev, S. (2000): *Conditional* Value-at-Risk*: Optimization Algorithms and Applications.* Financial Engineering News, No. 14, February.
(http://uryasev.ams.stonybrook.edu/wp-content/uploads/2011/11/FinNews.pdf)

RISK MANAGEMENT AND PORTFOLIO OPTIMIZATION

Constraints on risk:

- **MaxDD:** $\mathbf{M}(x) \le \nu_1$
- **AvDD:** $\mathbf{A}(x) \le \nu_2$
- **CDaR:** $\Delta_\alpha(x) \le \nu_3$

where ν_1, ν_2, ν_3 **are fractions of the initial investment, and**

$$0 \le \nu_1, \nu_2, \nu_3 \le 1$$

- **Combinations:** $\mathbf{M}(x) \le \nu_1,\ \mathbf{A}(x) \le \nu_2,\ \Delta_\alpha(x) \le \nu_3$

Optimization problems:

Maximization of the portfolio expected cumulative rate of return at the final moment T **subject to constraints on risk**

$$\max_x E\,\left[r(x,T,\omega)\right] \quad s.\ t.\ \mathbf{M}(x) \le \nu_1, \quad x \in X$$

$$\max_x E\,\left[r(x,T,\omega)\right] \quad s.\ t.\ \mathbf{A}(x) \le \nu_2, \quad x \in X$$

$$\max_x E\,\left[r(x,T,\omega)\right] \quad s.\ t.\ \Delta_\alpha(x) \le \nu_3, \quad x \in X$$

where X **is the set of "technological" constraints.**

A REAL-LIFE PORTFOLIO OPTIMIZATION EXAMPLE

How to allocate resources among different proprietary investment strategies to maximize portfolio value at the final moment while constraining portfolio drawdown? Only one historical sample-path is available.

$m = 32,$ **number of different proprietary investment strategies**

$C = \$20\text{M},$ **amount of the initial investment**

$N = 3180,$ **number of working days (one working day is a time unit)**

$d = 12$ **years**, **time horizon**

- **Cumulative uncompounded rates of return for all instrument at each time interval**

$$r_j = (r_{j1}, r_{j2}, \ldots r_{jk}, \ldots r_{jm}), \ j = \overline{1, N}.$$

- **Uncompounded cumulative portfolio rate of return at time moment** j

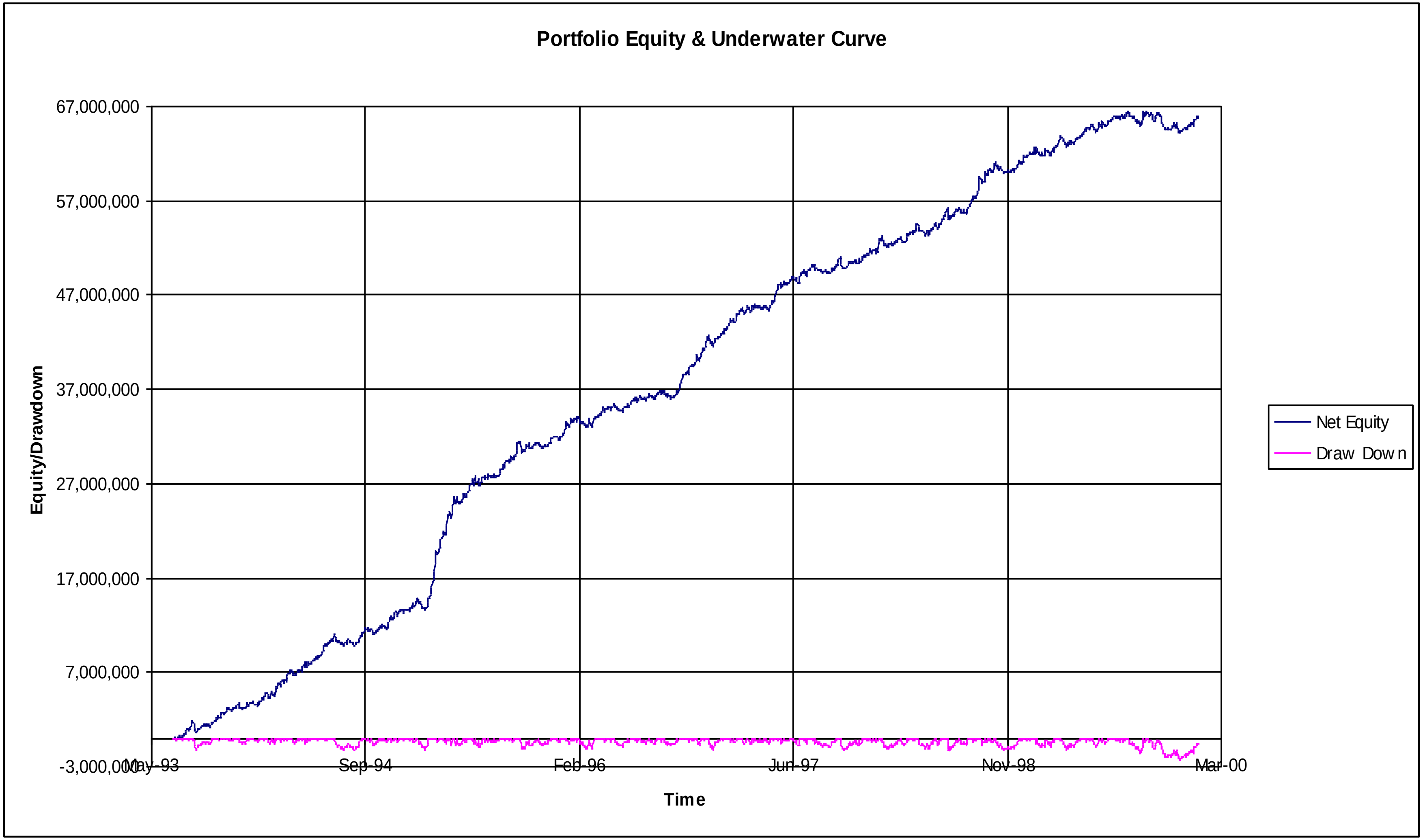
$$r(x, j) = r_j x = \sum_{k=1}^{m} r_{jk} x_k .$$

- **Technological constraints are given in the form**

$$X = \left\{ \ x : x_{\min} \le x_k \le x_{\max}, \ \forall \ \ k = \overline{1, m} \ \right\}$$

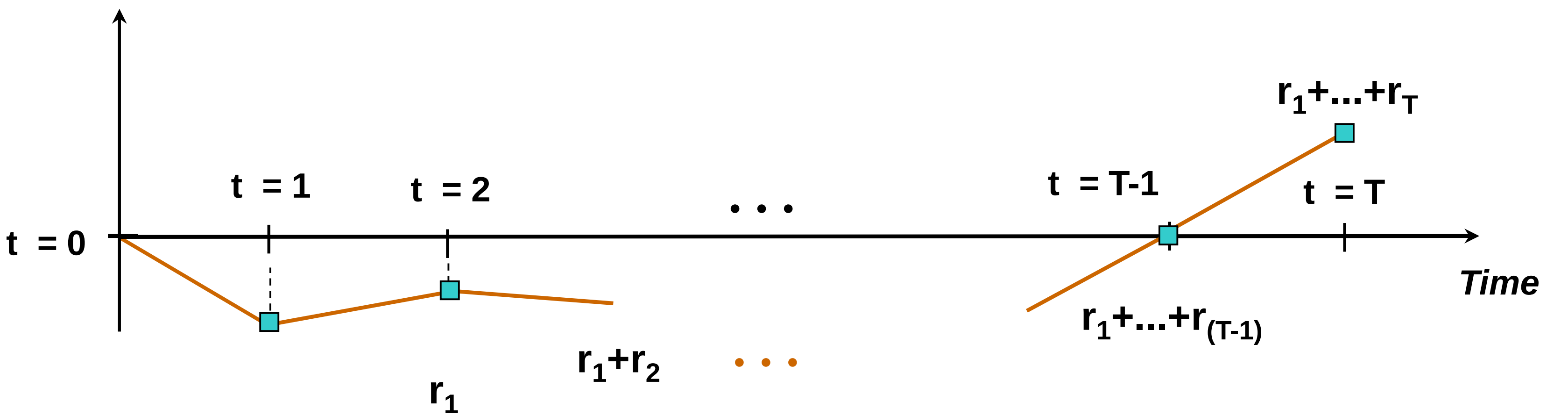
$$x_{\min} = 0.2, \ \ x_{\max} = 0.8.$$

PORTFOLIO VALUE & UNDERWATER CURVE

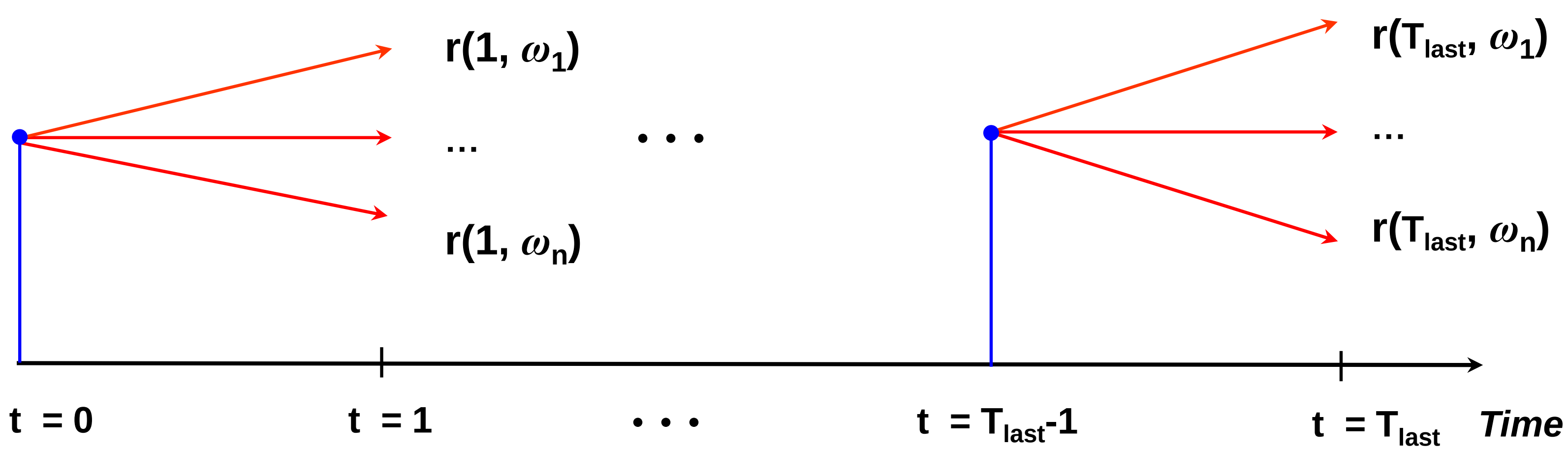


STOCHASTIC MODEL

Historical Uncompounded Cumulative Portfolio Rate of Return



Sequence of trials (bootstrap simulation)



OPTIMIZATION PROBLEMS

- **One scenario: historical equity curve**
- **Optimization problems are solved with Portfolio Safeguard, which has a precoded CDaR function**

Constraint on CDaR (similar constraints for MaxDD and for AvDD)

$$\max_{x} \left\{ \tfrac{1}{d} r_N \cdot x \right\}$$

$$\zeta + \frac{1}{(1-\alpha)N} \sum_{i=1}^{N} \left(\{ \max_{1 \le j \le i} \{ r_j \cdot x \} - r_i \cdot x \} - \zeta \right)^+ \le \nu_3$$

$$x_{\min} \le x_k \le x_{\max}, \ \forall \ k = \overline{1, m}$$

$$(g)^+ = \max\{0, g\}$$

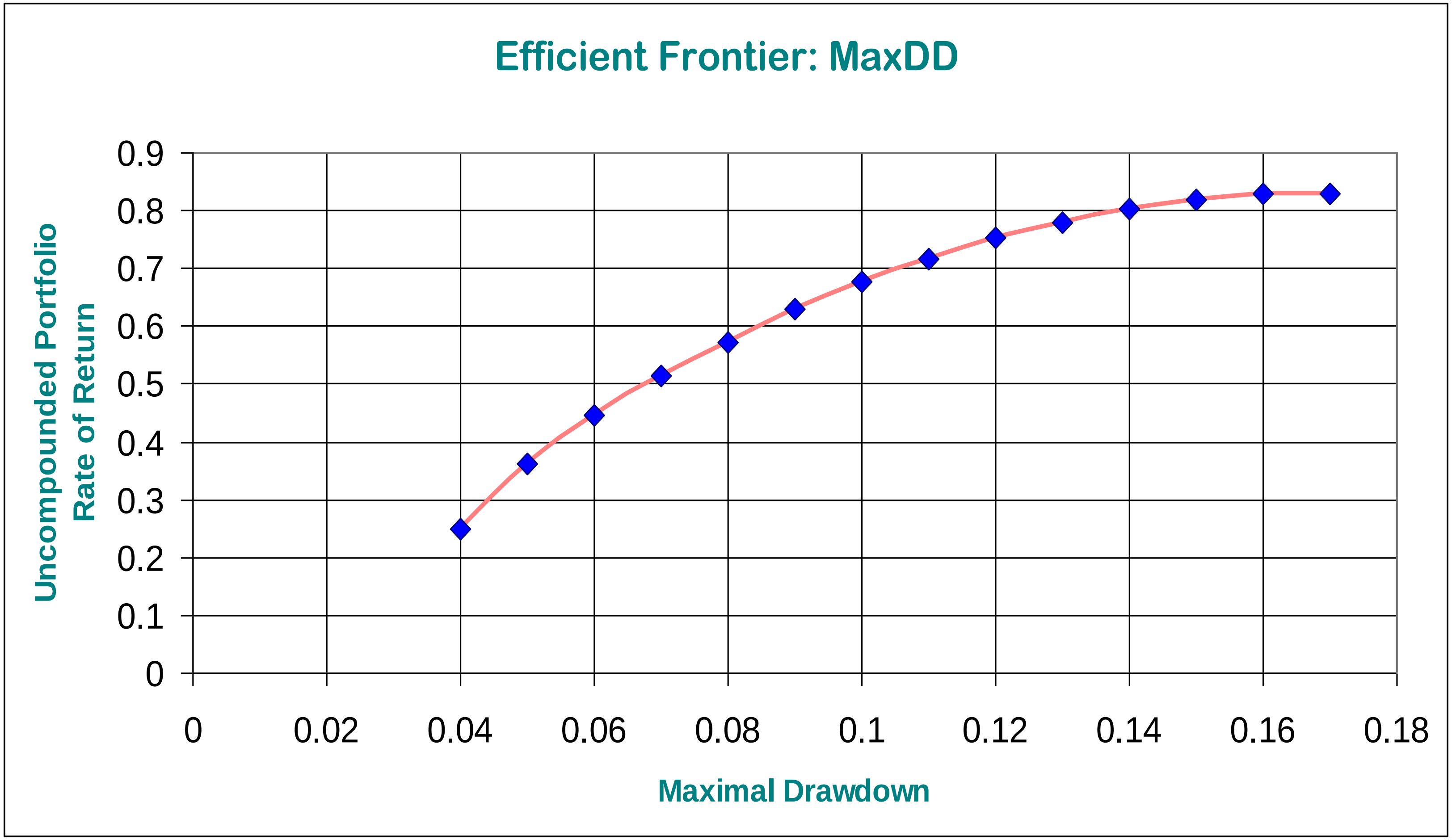
MaxDD SOLUTION

Risk	4.0%	5.0%	6.0%	7.0%	8.0%	9.0%	10.0%	11.0%	12.0%	13.0%	14.0%	15.0%	16.0%	17.0%
Rate of Return	25.0%	36.3%	44.5%	51.4%	57.3%	63.0%	67.7%	71.7%	75.2%	78.0%	80.4%	81.9%	82.9%	83.0%
AD	0.200	0.397	0.740	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800
US	0.200	0.200	0.200	0.200	0.618	0.409	0.532	0.560	0.800	0.800	0.800	0.800	0.800	0.800
BP	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.222	0.512	0.768	0.800	0.800	0.800
CD	0.251	0.592	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800
HG	0.625	0.800	0.768	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800
DX	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.629	0.800	0.800	0.800	0.800	0.800
ED	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.349	0.741	0.800	0.800	0.800
EC	0.200	0.200	0.200	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800
FXADJY	0.270	0.576	0.771	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800
FXBPJY	0.200	0.200	0.200	0.200	0.200	0.200	0.533	0.800	0.800	0.800	0.800	0.800	0.800	0.800
FXEUBP	0.200	0.280	0.294	0.320	0.337	0.651	0.723	0.800	0.800	0.800	0.800	0.800	0.800	0.800
FXEUJY	0.200	0.200	0.405	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800
FXEUSF	0.328	0.200	0.254	0.298	0.729	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800
FXNZUS	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.266	0.800	0.800
FXUSSG	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.281	0.212	0.434	0.716	0.800	0.800	0.800
FXUSSE	0.200	0.800	0.800	0.652	0.732	0.704	0.598	0.350	0.200	0.200	0.200	0.800	0.800	0.800
FV	0.200	0.200	0.387	0.581	0.519	0.503	0.541	0.800	0.800	0.800	0.800	0.800	0.800	0.800
GC	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.566	0.800	0.800
JY	0.200	0.228	0.336	0.250	0.372	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800
LFT	0.200	0.200	0.200	0.200	0.385	0.634	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800
LGL	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.373	0.800	0.800	0.800	0.800	0.800	0.800
LBT	0.200	0.350	0.623	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800
LAHD	0.200	0.271	0.365	0.457	0.510	0.597	0.780	0.800	0.800	0.800	0.800	0.800	0.800	0.800
MNN	0.200	0.300	0.418	0.450	0.437	0.800	0.800	0.800	0.770	0.800	0.800	0.800	0.800	0.800
SF	0.200	0.200	0.366	0.393	0.519	0.520	0.633	0.751	0.800	0.800	0.800	0.800	0.800	0.800
AAO	0.200	0.247	0.255	0.277	0.210	0.394	0.677	0.800	0.685	0.800	0.800	0.800	0.800	0.800
AXB	0.200	0.374	0.320	0.469	0.630	0.800	0.551	0.636	0.800	0.800	0.800	0.800	0.800	0.800
SI	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.396	0.800
SJB	0.492	0.742	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800
SNI	0.200	0.560	0.673	0.686	0.781	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800
TY	0.200	0.200	0.225	0.320	0.601	0.688	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800
DGB	0.200	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800	0.800

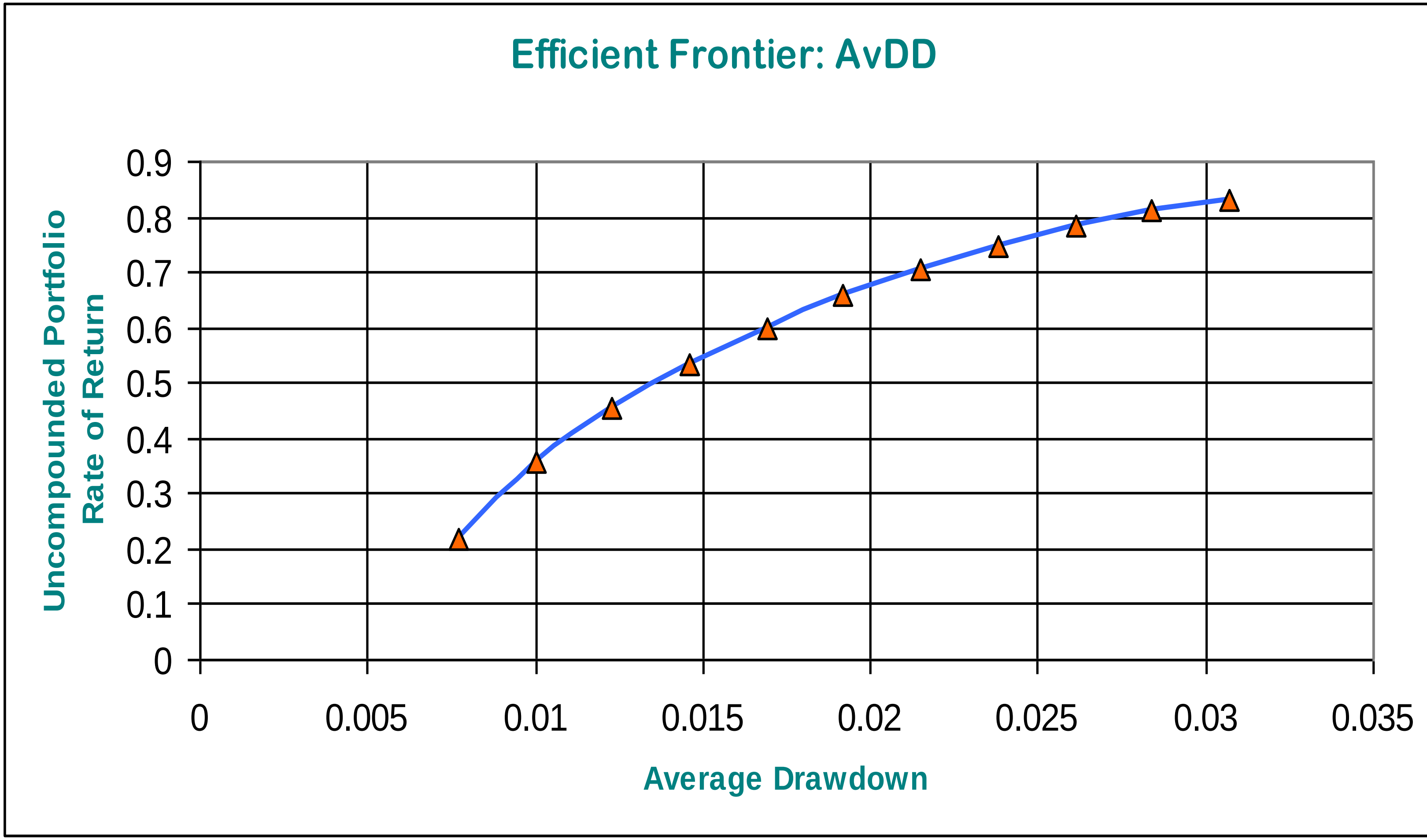
AvDD SOLUTION

Risk	0.8%	0.9%	1.0%	1.5%	2.0%	2.5%	3.0%	3.5%
Rate of Return	25.1%	30.9%	35.6%	54.5%	67.4%	76.7%	82.7%	83.0%
AD	0.298	0.457	0.572	0.800	0.800	0.800	0.800	0.800
US	0.200	0.200	0.200	0.200	0.200	0.405	0.800	0.800
BP	0.200	0.200	0.200	0.200	0.200	0.666	0.800	0.800
CD	0.256	0.309	0.373	0.800	0.800	0.800	0.800	0.800
HG	0.416	0.481	0.597	0.800	0.800	0.800	0.800	0.800
DX	0.200	0.200	0.200	0.200	0.200	0.200	0.800	0.800
ED	0.200	0.204	0.297	0.323	0.378	0.725	0.800	0.800
EC	0.200	0.200	0.200	0.200	0.615	0.800	0.800	0.800
FXADJY	0.200	0.200	0.200	0.331	0.602	0.800	0.800	0.800
FXBPJY	0.200	0.200	0.200	0.501	0.800	0.800	0.800	0.800
FXEUBP	0.200	0.200	0.200	0.578	0.800	0.800	0.800	0.800
FXEUJY	0.200	0.342	0.590	0.800	0.800	0.800	0.800	0.800
FXEUSF	0.398	0.557	0.622	0.800	0.800	0.800	0.800	0.800
FXNZUS	0.200	0.200	0.200	0.200	0.800	0.800	0.800	0.800
FXUSSG	0.200	0.200	0.200	0.476	0.800	0.800	0.800	0.800
FXUSSE	0.246	0.513	0.744	0.800	0.800	0.800	0.800	0.800
FV	0.200	0.200	0.200	0.525	0.800	0.800	0.800	0.800
GC	0.200	0.200	0.200	0.200	0.200	0.200	0.467	0.800
JY	0.200	0.292	0.378	0.800	0.800	0.800	0.800	0.800
LFT	0.200	0.200	0.200	0.200	0.200	0.313	0.800	0.800
LGL	0.200	0.200	0.200	0.200	0.425	0.773	0.800	0.800
LBT	0.200	0.389	0.522	0.800	0.800	0.800	0.800	0.800
LAHD	0.200	0.200	0.200	0.337	0.665	0.800	0.800	0.800
MNN	0.200	0.200	0.200	0.200	0.679	0.800	0.800	0.800
SF	0.200	0.200	0.200	0.539	0.800	0.800	0.800	0.800
AAO	0.229	0.402	0.462	0.800	0.800	0.800	0.800	0.800
AXB	0.200	0.200	0.200	0.620	0.800	0.800	0.800	0.800
SI	0.200	0.200	0.200	0.200	0.200	0.200	0.546	0.800
SJB	0.384	0.547	0.669	0.800	0.800	0.800	0.800	0.800
SNI	0.200	0.292	0.331	0.657	0.800	0.800	0.800	0.800
TY	0.200	0.200	0.200	0.200	0.200	0.721	0.800	0.800
DGB	0.665	0.800	0.800	0.800	0.800	0.800	0.800	0.800

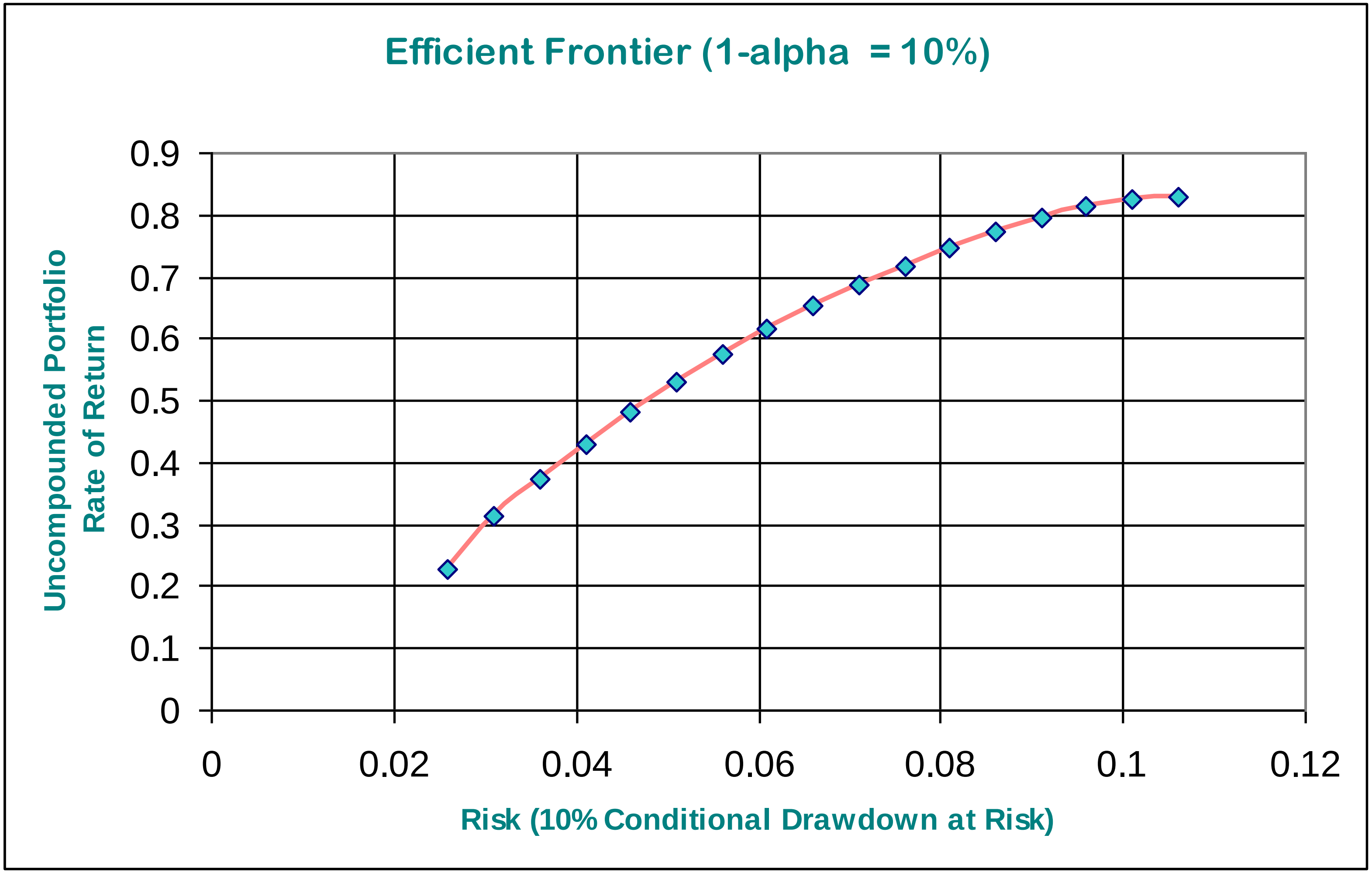
MaxDD EFFICIENT FRONTIER



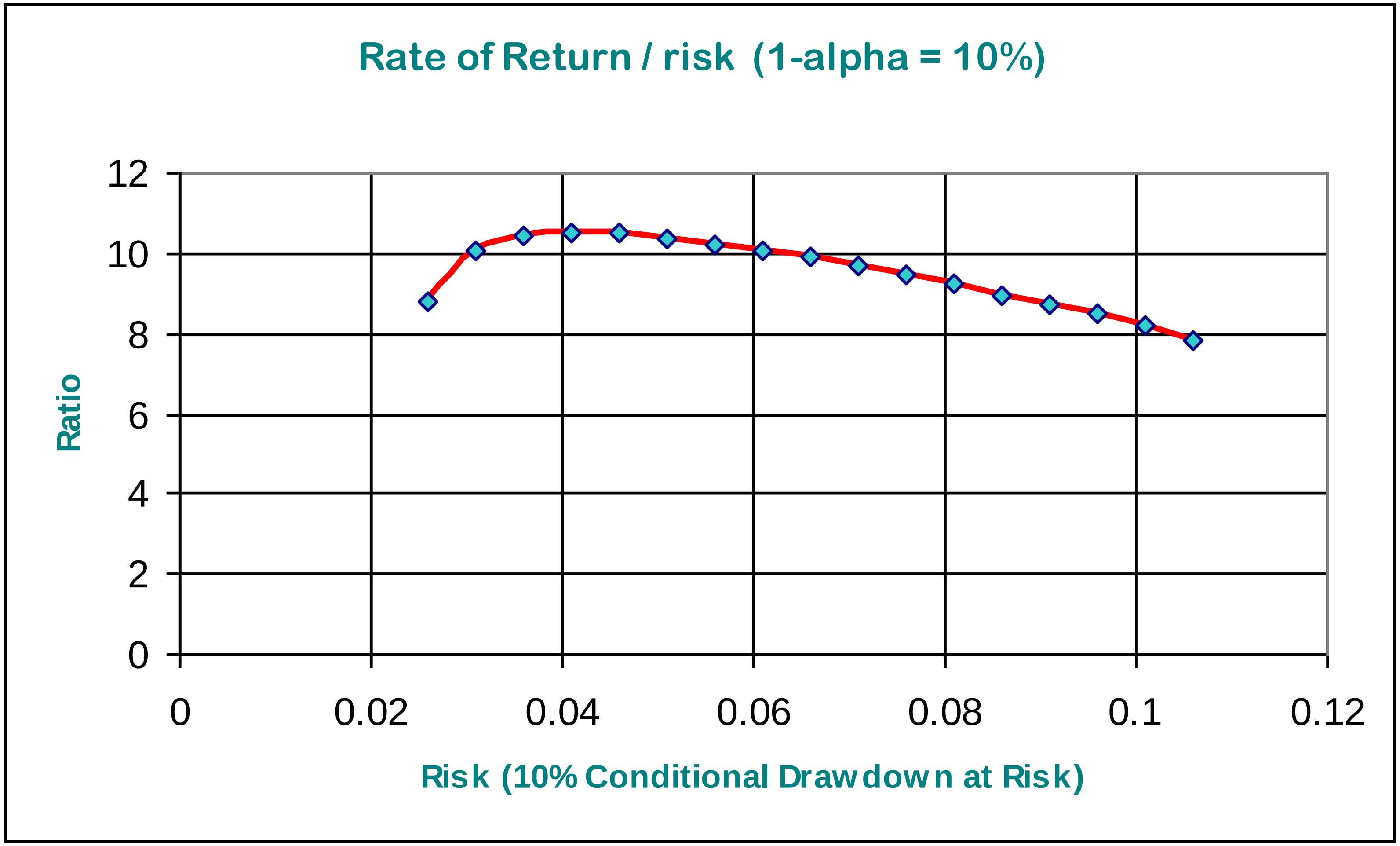
AvDD EFFICIENT FRONTIER



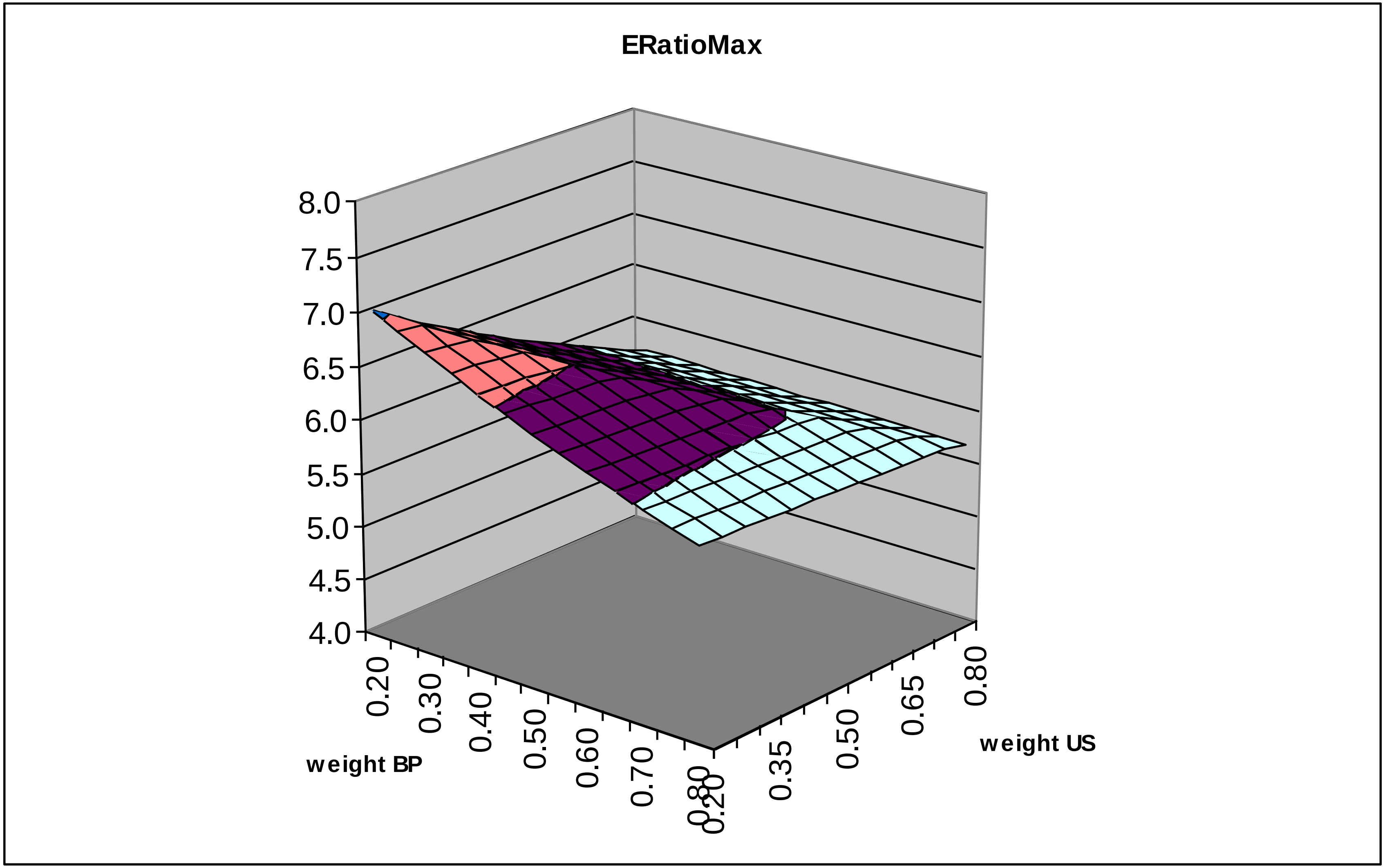
CDaR EFFICIENT FRONTIER



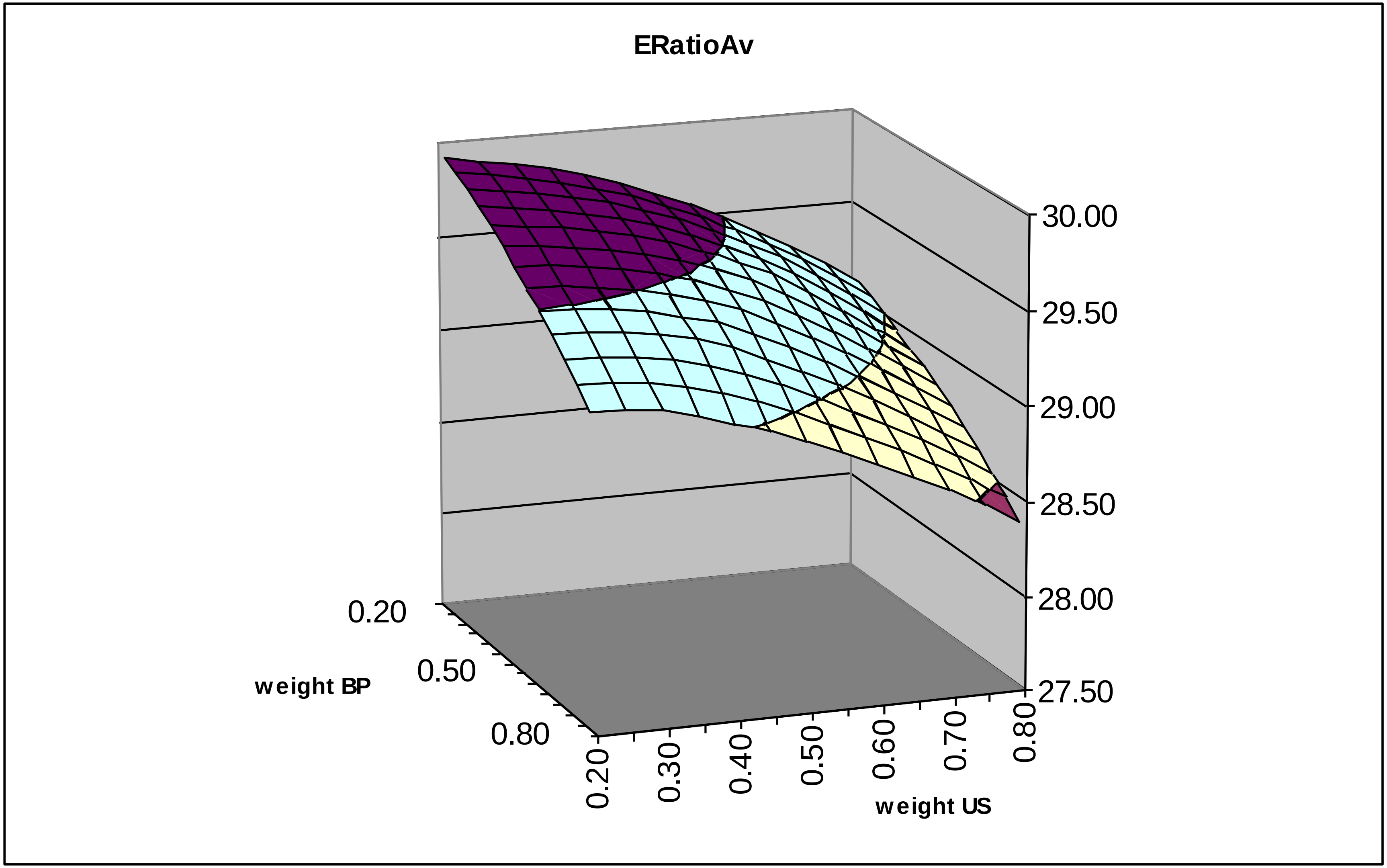
RATE OF RETURN / RISK RATIO (10% CDaR)



REWARD / RISK RATIOS - MaxDD



REWARD / RISK RATIOS - AvDD



Case Studies

- **Portfolio Optimization with Drawdown Constraints on a Single Path**

http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/portfolio- optimization-with-drawdown-constraints-on-a-single-path/

- **Portfolio Optimization with Drawdown Constraints on Multiple Paths**

http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/portfolio-optimization-with-drawdown-constraints-on-multiple-paths/

- **Portfolio Optimization with Drawdown Constraints, Single Path vs Multiple Paths**

http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/portfolio-optimization-with-drawdown-constraints-single-path-vs-multiple-paths/

- **CoCDaR and mCoCDaR Approaches: Risk Contribution Measurement**

http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/case-study-cocdar-approach-systemic-risk-contribution-measurement/

- **Style Classification with mCoCDaR Regression**

http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/case-study-style-classification-with-mcocdar-regression/

CONCLUSION

- **Introduced CDaR family of risk measures based on a notion of drawdown (underwater) curve**
- **Portfolio allocation problems are reduced to convex (linearizable) programming problems: efficient algorithms**
- **Solved a particular real-life example on the basis of historical equity curves**
- **CDaR portfolio optimization strategy generates stable portfolio allocations**