High-Frequency Trading and Modern Market Microstructure

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Parts of this talk are joint work with
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A Simplified View of Trading
A Simplified View of Trading

portfolio manager (buy-side)
A Simplified View of Trading

- Portfolio manager (buy-side)
- Algorithmic trading engine (buy- or sell-side)

Exchanges:
- ARCA
- NASDAQ
- BATS
- Dark Pool #1

Market makers / high-frequency traders
A Simplified View of Trading

portfolio manager (buy-side)

algorithmic trading engine (buy- or sell-side)

market centers

ARCA  NASDAQ  BATS  …  Dark Pool #1  …
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ARCA  NASDAQ  BATS  ...  Dark Pool #1  ...

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Modern U.S. Equity Markets

Electronic
Decentralized / Fragmented
NYSE, NASDAQ, ARCA, BATS, Direct Edge, …
Exchanges (≈ 70%)
electronic limit order books
Alternative venues (≈ 30%)
ECNs, dark pools, internalization, OTC market makers, etc.
Participants increasingly automated
investors: "algorithmic trading"
market makers: "high-frequency trading"
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Algorithmic Trading

Algorithmic trading of a large order is typically decomposed into three steps:

1. **Trade Scheduling:**
   - Splits parent order into \( \sim 5 \) min "slices".
   - Relevant time-scale: minutes-hours.
   - Tradeoff time with execution costs.
   - Reflects price impact (temporary / permanent).
   - Reflects urgency, "alpha," risk/return.
   - Schedule updated during execution to reflect price/liquidity/…

2. **Optimal Execution of a Slice:**
   - Divides slice into child orders.
   - Relevant time-scale: seconds–minutes.
   - Strategy optimizes pricing and placing of orders in the limit order book.
   - Tradeoff of price versus delay.
   - Execution adjusts to speed of LOB dynamics, price momentum, …

3. **Order Routing:**
   - Decides where to send each child order.
   - Relevant time-scale: \( \sim 1 \)–50 ms.
   - Optimizes fee/rebate tradeoff, liquidity/price, latency, etc.
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Market-Making

- Supply short-term liquidity and capture the bid-ask spread
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- Critical to model **adverse selection**: short-term price change conditional on a trade, “order flow toxicity”
  
  profit of a single trade \( \approx (\text{captured spread}) - (\text{adverse selection}) \)
  
  depends on volatility, news, market venue, counterparty, etc.
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- Strategies very sensitive to microstructure details of market mechanisms
Relevant Questions: Participants

Decision problems: Algorithmic trading

- How to schedule trades over time?
- Which market mechanisms to use? Limit order, market order, dark pool? At what price?
- How to route trades across venues?
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Decision problems: Market-making

- If and how much liquidity to supply?
- At what price?
- What market venues / mechanisms to use?
- When to aggressively trade ahead of adverse price movements?
Relevant Questions: Participants

Estimation/prediction problems:

- Pre-trade analytics (e.g., execution costs)
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- Pre-trade analytics (e.g., execution costs)
- Short-term price changes typically statistically weak (e.g., $R^2 \ll 1\%$)
  non-stationary

Interdependencies between prices and trades
price impact, adverse selection
often need to use own trading data
significantly endogeneity issues
Other market primitives (e.g., volume, liquidity, volatility, …)
Other microstructure primitives (e.g., order fill probabilities, order completion times)
Post-trade analytics (measure / attribute performance)
Relevant Questions: Participants

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completion times)
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Computational / implementation challenges:

- **Big data**
  100+ GB per day (compressed!) for U.S. equities
  parallelism / memory efficiency important

- **Realtime, low latency decision making**
  down to microsecond time scales
  linear algebra is possible (e.g., Kalman filter update)
  but how about optimization (e.g., mean-variance QP)?
  how to implement a complex control policy?
  decompose across CPUs, across time scales?
  use exotic hardware (e.g., FPGA’s)?
The May 6, 2010 Flash Crash

SPY Volume and Price

(Source: Joint CTFC SEC Report, 9/30/2010)
Relevant Questions: Regulators

How to ensure robust / fair markets:

- Are markets more or less efficient than in the past?
- Why do we need to trade on a millisecond timescale?
- Should we have such a fragmented market?
- Are market structures like dark pools beneficial?
There are many interesting and important open questions — there is room for innovation in modeling / problem formulation as well as methodology.
Outline

There are many interesting and important open questions — there is room for innovation in modeling / problem formulation as well as methodology.

We will consider in detail a handful of specific problems:

- The Cost of Latency
- Order Routing and Fragmented Markets
- Dark Pools
The Cost of Latency

Joint work with Mehmet Sağlam.
Powerful computers, some housed right next to the machines that drive marketplaces like the New York Stock Exchange, enable high-frequency traders to transmit millions of orders at lightning speed.

…

High-frequency traders often confound other investors by issuing and then canceling orders almost simultaneously … And their computers can essentially bully slower investors into giving up profits.

…

“It’s become a technological arms race, and what separates winners and losers is how fast they can move,” said Joseph M. Mecane of NYSE Euronext, which operates the New York Stock Exchange.
Latency and Its Significance

Introduction to latency:
- Delay between a trading decision and its implementation

Driven by technological innovation and competition between exchanges

Why is low-latency trading important for market participants?
- Contemporaneous decision making
- Competitive advantage/disadvantage
- Time priority rules in microstructure
Latency and Its Significance

Introduction to latency:

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- 2 minutes (NYSE, pre-1980)
  ⇒ 20 seconds (NYSE, 1980)
  ⇒ 100s of milliseconds (NYSE, 2007)
  ⇒ 1 millisecond (NYSE Arca, 2009) “low latency”
  ⇒ 10–100 microseconds (current state-of-the-art) “ultra-low latency”
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Why is low-latency trading important for market participants?

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- Time priority rules in microstructure
Our Contributions

- **Quantify** the cost of latency in a benchmark stylized execution problem
- Provide a *closed-form* expression in well-known market parameters
- Latency is important to all investors with low cost structures
Literature Review

- Empirical work on the effects of improvements in trading technology
  [Easley, Hendershott, Ramadorai, 2008]
  [Hendershott, Jones, Menkveld, 2008]
  [Hasbrouck & Saar, 2008, 2010; Stoikov, 2009; Kearns, et al., 2010; others]

- Latency of the price ticker
  [Cespa & Foucault, 2007]

- Optionality embedded in limit orders
  [Angel, 1994; Harris, 1998; others]
  [Copeland & Galai, 1983; Chacko, Jurek, Stafford, 2008; others]

- Discrete-time hedging of contingent claims
  [Boyle & Emanuel, 1980; Bertsimas, Kogan, Lo, 2000; others]
Benchmark Model: No Latency

- An investor must sell 1 share of stock over the time horizon $[0, T]$
Benchmark Model: No Latency

- An investor must sell 1 share of stock over the time horizon \([0, T]\)
- Option A: place a market order to sell
  \[ S_t = \text{bid price at time } t, \quad S_t \sim S_0 + \sigma B_t \]
An investor must sell 1 share of stock over the time horizon $[0, T]$.

Option A: place a market order to sell

$$S_t = \text{bid price at time } t, \quad S_t \sim S_0 + \sigma B_t$$

Option B: place a limit order to sell

$$L_t = \text{limit order price at time } t \text{ (decision variable)}$$
A limit order executes if either:

- Market buy order arrives and 
  \[ L_t \leq S_t + \delta \]
- Impatient buyers arrive, Poisson \((\mu)\) limit order executes

\[ S_{\tau_1} + \delta_{\tau_1} \]
\[ S_{\tau_2} + \delta_{\tau_2} \]

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Benchmark Model: No Latency

Objective: maximize $E[\text{sale price}] - S_0$
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Optimal strategy: ‘Pegging’

- Use limit orders, $L_t = S_t + \delta$
- If not executed by time $T$, sell with a market order at $S_T$

$\tau =$ arrival time of next impatient buyer

Value $= E \left[ S_{\tau \wedge T} + \delta \mathbb{I}_{\{\tau \leq T\}} \right] - S_0$

$= \delta \left( 1 - e^{-\mu T} \right)$
Latency Model

\[ \Delta t = \text{Latency} \]

\[ T_0 = 0 \quad \cdots \quad T_i = i\Delta t \quad T_{i+1} \quad T_{i+2} \quad \cdots \quad T = n\Delta t \]
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\[ h_i(\Delta t) \triangleq \sup_{\ell_i, \ell_{i+1}, \ldots} E[\text{sale price} | \mathcal{F}_{T_i}, \text{no trade in } [0, T_{i+1})] - S_{T_i} \]
Lemma. (Consistency)

\[
\lim_{\Delta t \to 0} h_0(\Delta t) = \delta \left( 1 - e^{-\mu T} \right)
\]

\[
\bar{h}_0
\]
Main Result

Lemma. (Consistency)
\[
\lim_{\Delta t \to 0} h_0(\Delta t) = \delta \left( 1 - e^{-\mu T} \right) \\
\]

Theorem. (Asymptotic Value)
As \( \Delta t \to 0 \),
\[
h_0(\Delta t) = \bar{h}_0 \cdot \left( 1 - \frac{\sigma}{\delta} \sqrt{\Delta t \log \frac{\delta^2}{2\pi \sigma^2 \Delta t}} \right) + o \left( \sqrt{\Delta t} \right)
\]
Latency Cost

Definition.

Latency Cost $\triangleq \frac{\bar{h}_0 - h_0(\Delta t)}{\bar{h}_0}$

Interpretation: The cost of latency as a percentage of ‘cost of immediacy’ in the absence of any latency

Corollary: As $\Delta t \to 0$, Latency Cost $= \sigma \sqrt{\Delta t} \delta \sqrt{\log \delta} / \pi \sigma^2 \Delta t + o(\sqrt{\Delta t})$

Does not depend on $\mu$ or $T$

Increasing function of $\sigma \sqrt{\Delta t} / \delta$

Increasing marginal benefits to reductions in latency
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- Does not depend on \( \mu \) or \( T \)
- Increasing function of \( \sigma \sqrt{\Delta t}/\delta \)
- Increasing marginal benefits to reductions in latency
Empirical Application: GS Latency Cost

- Parameters estimated for Goldman Sachs Group, Inc., January 2010
Interpretation

- Normalize cost of immediacy to $0.01 per share
- Value of decreasing latency from 500 ms to 1 ms: $0.0020
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- Value of decreasing latency from 500 ms to 1 ms: $\approx 0.0020$

- Typical high frequency trading profits: $0.0010 – 0.0020$

  sources: Tradeworx, Inc.; Knight Trading 2009 10K filing; AQR Capital;
  Traders Magazine 11/2005, Q&A With Dave Cummings; etc.
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• Other trading costs for a large investor: $0.0005–$0.0015
  
  (e.g., brokerage & SEC fees, etc.)
Conclusion

Contributions:

- Quantify the cost of latency in a benchmark stylized execution problem
- Provide a closed-form expression in well-known market parameters

Implications:

- Latency can be important to any investor using limit orders
- How important depends on the rest of the investor’s trading costs (commissions, etc.)

Extensions:

- Empirical analysis of the historical evolution of latency cost and implied latency for U.S. equities
- More complex price dynamics (jump diffusions) and fill models
Order Routing and Fragmented Markets

Joint work with Costis Maglaras and Hua Zheng.
Motivation / Contributions

How to analyze fragmented LOBs?

- each a multi-class queueing system with complex dynamics
- in addition, agents exert control via “smart order routing”
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We propose a tractable model to analyze decentralized LOBs, incorporates:

- limit order routing capability
- market order routing capability
- time-money tradeoff heterogeneity
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In this framework, we:

- characterize the equilibrium through a fluid model
- establish that routing decisions simplify dynamics: state space collapse
- provide empirical findings supporting state space collapse
Related Literature

- Market microstructure:
  Kyle; Glosten-Milgrom; Glosten; …

- Empirical analysis of limit order books:
  Bouchaud et al.; Hollifield et al.; Smith et al.; …

- Dynamic models and optimal execution in LOB:
  Obizhaeva & Wang; Cont, Stoikov, Talreja; Rosu; Alfonsi et al.; Foucault et al.; Parlour; Stoikov, Avellaneda, Reed; Cont & de Larrard; Predoiu et al.; Guo & de Larrard …

- Transaction cost modeling, adverse selection, …: Madhavan; Dufour & Engle; Holthausen et al.; Huberman & Stanzl; Almgren et al.; Gatheral; Sofianos; …

- Make/take fees, liquidity cycles, etc.: Foucault et al.; Malinova & Park

- Stochastic models of multi-class queueing networks
The Limit Order Book (LOB)
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- Buy limit order arrivals
- Cancellations
- Market sell orders
- Market buy orders
- Sell limit order arrivals
- Cancellations
We consider the evolution of:

- one side of the market
- the `top-of-the-book', i.e., national best bid queues across all exchanges

national best bid/ask (NBBO)
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1. one side of the market
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Multiple Limit Order Books

national best bid/ask (NBBO)
Three relevant time scales:

- **Events:** order / trade / cancellation interarrival times \( \sim \) ms – sec
- **Delays:** waiting times at different exchanges \( \sim \) sec – min
- **Rates:** time-of-day variation of flow characteristics \( \sim \) min – hrs
Time Scales

Three relevant time scales:

- **Events**: order / trade / cancellation interarrival times  
  \( \sim \) ms – sec

- **Delays**: waiting times at different exchanges  
  \( \sim \) sec – min

- **Rates**: time-of-day variation of flow characteristics  
  \( \sim \) min – hrs

Order placement decisions depend on queueing delays in LOBs (our focus)

- assume constant arrival rates of limit orders and trades
- order sizes are small relative to overall flow over relevant time scale
- overall limit order and trade volumes are high
One-sided Multi LOB Fluid Model

**Fluid model:** Continuous & deterministic arrivals of infinitesimal traders
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Related to: Exchange 1

Related to: Exchange 2

Related to: Exchange $N$
One-sided Multi LOB Fluid Model

**Fluid model:** Continuous & deterministic arrivals of infinitesimal traders
**One-sided Multi LOB Fluid Model**

**Fluid model:** Continuous & deterministic arrivals of infinitesimal traders

![Diagram of fluid model](image)

- Continuous & deterministic arrivals of infinitesimal traders
- Market order
- Exchange 1
- Exchange 2
- Exchange N
- Dedicated limit order flow
- Optimized limit order flow
One-sided Multi LOB Fluid Model

Fluid model: Continuous & deterministic arrivals of infinitesimal traders

\[
\lambda_1 \rightarrow \Lambda \rightarrow \text{market order} \rightarrow \text{exchange 1} \leftarrow \mu_1
\]

\[
\lambda_2 \rightarrow \Lambda \rightarrow \text{market order} \rightarrow \text{exchange 2} \leftarrow \mu_2
\]

\[
\vdots
\]

\[
\lambda_n \rightarrow \Lambda \rightarrow \text{market order} \rightarrow \text{exchange } N \leftarrow \mu_N
\]

- dedicated limit order flow
- optimized limit order flow
- market order flow (attraction model)
The Limit Order Placement Decision

Factors affecting limit order placement:

- Expected delay ($\approx 1$ to $1000$ seconds)
- Rebates ($\approx -$0.0002 to -$0.0030 per share)

Traders choose to route their order to exchange $i$ given by

$$\arg\max_i \gamma r_i - ED_i$$

where $\gamma \sim F$ i.i.d. across traders, captures delay tolerance/rebate tradeoff. $\gamma \sim 10^{-1}$ to $10^4$ seconds per $\$0.01$ allows choice amongst Pareto efficient $(r_i, ED_i)$ pairs.

Implicit option for a market order: $r_0 \ll 0$, $ED_0 = 0$.33
The Limit Order Placement Decision

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\[ r_i = \text{rebate} \quad \text{ED}_i = \text{expected delay} \]
The Limit Order Placement Decision

Factors affecting limit order placement:

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Traders choose to route their order to exchange $i$ given by

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- $\gamma \sim F$ i.i.d. across traders, captures delay tolerance / rebate tradeoff
  \[ \Rightarrow \gamma \sim 10^1 \text{ to } 10^4 \text{ seconds per }$0.01
- allows choice amongst Pareto efficient $(r_i, \text{ED}_i)$ pairs
- Implicit option for a market order: $r_0 \ll 0$, $\text{ED}_0 = 0$
Market orders execute immediately, no queueing

Market orders incur fees ($\approx r_i$)

Natural criterion is to route an infinitesimal order according to

$$\text{argmin}_i \{ r_i : Q_i > 0, \ i = 1, \ldots, N \}$$

Routing decision differs from "fee minimization" due to:

1. Order sizes are not infinitesimal; may have to be split across exchanges
2. Latency to exchange introduces notion of $P(\ell)$ when $Q_i$ are small
3. Not all orders are "optimized", or have other economic considerations
4. Traders avoid "clearing" queues to avoid increased price slippage
The Market Order Routing Decision

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- Market orders incur fees (≈ \( r_i \))
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Routing decision differs from “fee minimization” due to

- Order sizes are not infinitesimal; may have to be split across exchanges
- Latency to exchange introduces notion of P(fill) when \( Q_i \) are small
- Not all flow is “optimized”, or has other economic considerations
- Traders avoid “clearing” queues to avoid increased price slippage
The Market Order Routing Decision

Attraction Model: Bounded rationality and model intricacies motivate fitting a probabilistic model of the form

\[ \mu_i(Q) = \mu \frac{f_i(Q_i)}{\sum_j f_j(Q_j)} \]

- \( f_i(\cdot) \) captures "attraction" of exchange \( i \):
  \[ \uparrow \text{ in } Q_i \text{ and } \downarrow \text{ in } r_i \]

- Remainder of this talk uses:
  \[ f_i(Q_i) = \beta_i Q_i \]
  (we imagine \( \beta_i \sim 1/r_i \))
Transient Dynamics & Flow Equilibrium

- **Dynamics:** Coupled ODEs describe $\dot{Q}(t)$ dynamics, depend on $(\lambda, \Lambda, \mu)$
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• **Equilibrium:** Fixed point $Q^*$
Dynamics: Coupled ODEs describe $\dot{Q}(t)$ dynamics, depend on $(\lambda, \Lambda, \mu)$

Equilibrium: Fixed point $Q^*$

Downstream analysis: (not in talk)
- $(\lambda, \Lambda, \mu)$ vary stochastically at slower time scale ($\sim$ min–hrs) than queue delays ($\sim$ sec–min)
- $Q$ evolves stochastically over such “steady state configurations”
- Point-wise stochastic fluid model
• **Dynamics:** Coupled ODEs describe $\dot{Q}(t)$ dynamics, depend on $(\lambda, \Lambda, \mu)$

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  • $Q$ evolves stochastically over such “steady state configurations”
  • Point-wise stochastic fluid model

We will analyze the structure of equilibria $Q^*$ as a function of $(\lambda, \Lambda, \mu)$. 
Fluid Model Equilibrium

\[ \pi_i(\gamma) = \text{fraction of type } \gamma \text{ investors who send orders to exchange } i \]

**Definition.** An equilibrium \((\pi^*, Q^*)\) must satisfy

(i) Individual rationality:

\[
\pi_i^*(\gamma) \ \text{optimizes} \ 
\max \pi_i(\gamma) \ 
\gamma \ r_0 + \sum_{i=1}^{\gamma} \pi_i(\gamma) (\gamma_r_i - Q_i^* \mu_i(Q^*))
\]

subject to

\[ \pi_i(\gamma) \geq 0 \]

\[ \sum_{i=1}^{\gamma} \pi_i(\gamma) = 1 \]

(ii) Flow balance:

\[ \lambda_i + \Lambda \int_0^{\infty} \pi_i^*(\gamma) dF(\gamma) = \mu_i(Q^*) \]
Fluid Model Equilibrium

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**Definition.** An equilibrium \((\pi^*, Q^*)\) must satisfy

(i) *Individual rationality:* for all \(\gamma\), \(\pi^*(\gamma)\) optimizes

\[
\text{maximize} \quad \pi_0(\gamma) \gamma r_0 + \sum_{i=1}^{N} \pi_i(\gamma) \left( \gamma r_i - \frac{Q_i^*}{\mu_i(Q^*)} \right)
\]

subject to \(\pi(\gamma) \geq 0, \sum_{i=0}^{N} \pi_i(\gamma) = 1\).
Fluid Model Equilibrium

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\]

subject to \(\pi(\gamma) \geq 0\), \(\sum_{i=0}^{N} \pi_i(\gamma) = 1\).

(ii) **Flow balance:** for all \(1 \leq i \leq N\),

\[
\lambda_i + \Lambda \int_0^{\infty} \pi_i^*(\gamma) \, dF(\gamma) = \mu_i(Q^*)
\]
Workload

- $W \equiv \sum_{i=1}^{N} \beta_i Q_i$ is the workload, a measurement of aggregate available liquidity.
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- \( \text{ED}_i = Q_i/\mu_i = (\sum_j \beta_j Q_j)/(\mu \beta_i) = W/(\mu \beta_i) \)
Workload

- \( W \triangleq \sum_{i=1}^{N} \beta_i Q_i \) is the \textit{workload}, a measurement of aggregate available liquidity

- \( W \neq \) total market depth, also accounts for time

- \( \text{ED}_i = Q_i / \mu_i = (\sum_j \beta_j Q_j) / (\mu \beta_i) = W / (\mu \beta_i) \)

- Workload is a sufficient statistic to determine delays
Fluid Model Equilibrium

Theorem. \((\pi^*, W^*)\) satisfy

\[
\begin{align*}
&\text{(i) Individual rationality:} \\
&\quad \text{for all } \gamma, \pi^*(\gamma) \text{ optimizes} \\
&\quad \max_{\gamma} \pi(\gamma) \int_0^\infty \left( \pi_0(\gamma) \gamma r_0 + \sum_{i=1}^N \pi_i(\gamma) (\gamma r_i - W^* \mu_{\beta i}) \right) dF(\gamma) \\
&\quad \text{subject to } \pi(\gamma) \geq 0, \sum_{i=1}^N \pi_i(\gamma) = 1. \\
\end{align*}
\]

\[
\begin{align*}
&\text{(ii) Systemic flow balance:} \\
&\quad \sum_{i=1}^N \left( \lambda_i + \Lambda \int_0^\infty \pi^*_i(\gamma) dF(\gamma) \right) = \mu \text{ if and only if } (\pi^*, Q^*) \text{ is an equilibrium, where} \\
&\quad Q^*_i \equiv \left( \lambda_i + \Lambda \int_0^\infty \pi^*_i(\gamma) dF(\gamma) \right) W^* \mu_{\beta i}
\end{align*}
\]
**Fluid Model Equilibrium**

**Theorem.** $(\pi^*, W^*)$ satisfy

(i) *Individual rationality:* for all $\gamma$, $\pi^*(\gamma)$ optimizes

$$\text{maximize } \int_0^\infty \left( \pi_0(\gamma) \gamma r_0 + \sum_{i=1}^N \pi_i(\gamma) \left( \gamma r_i - \frac{W^*}{\mu \beta_i} \right) \right) dF(\gamma)$$

subject to $\pi(\gamma) \geq 0$, $\sum_{i=0}^N \pi_i(\gamma) = 1$. 

(ii) *Systemic flow balance:* 

$$\sum_{i=1}^N \pi_i(\gamma) = 1$$

if and only if $(\pi^*, Q^*)$ is an equilibrium, where $Q^*_i \equiv \left( \lambda_i + \Lambda \int_0^\infty \pi^*_i(\gamma) dF(\gamma) \right)$.
Fluid Model Equilibrium

**Theorem.** \((\pi^*, W^*)\) satisfy

(i) *Individual rationality:* for all \(\gamma\), \(\pi^*(\gamma)\) optimizes

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\]

subject to \(\pi(\gamma) \geq 0, \sum_{i=0}^N \pi_i(\gamma) = 1\).

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\]

if and only if \((\pi^*, Q^*)\) is an equilibrium, where

\[
Q_i^* \equiv \left( \lambda_i + \Lambda \int_0^\infty \pi_i^*(\gamma) dF(\gamma) \right) \frac{W^*}{\mu \beta_i}
\]
Fluid Model Equilibrium

\[ W \propto \mu F^{-1}(\mu/\Lambda) \quad \text{[} \lambda_i = 0 \text{]} \]
Equilibrium Policies

\[
\min_i \gamma r_i - \frac{W^*}{\mu \beta_i}
\]
Empirical Results

- Consolidated feed TAQ data, millisecond timestamps
- Dow 30 Stocks; September 2011
- 6 largest exchanges for U.S. equities

<table>
<thead>
<tr>
<th>Exchange Code</th>
<th>Rebate ($ per share, \times 10^{-4})</th>
<th>Fee ($ per share, \times 10^{-4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>BATS</td>
<td>Z</td>
<td>27.0</td>
</tr>
<tr>
<td>DirectEdge X (EDGX)</td>
<td>K</td>
<td>23.0</td>
</tr>
<tr>
<td>NYSE ARCA</td>
<td>P</td>
<td>21.0†</td>
</tr>
<tr>
<td>NASDAQ OMX</td>
<td>T</td>
<td>20.0†</td>
</tr>
<tr>
<td>NYSE</td>
<td>N</td>
<td>17.0</td>
</tr>
<tr>
<td>DirectEdge A (EDGA)</td>
<td>J</td>
<td>5.0</td>
</tr>
</tbody>
</table>
Expected Delays: PCA Analysis

Under our model,

\[ \mathbf{ED}_{i,t} = \frac{Q_{i,t}}{\mu_{i,t}} = \frac{W_t}{\mu_t} \cdot \frac{1}{\beta_i} \]

Therefore, the vector of expected delays

\[ \vec{ED}_t \triangleq \left( \frac{Q_{1,t}}{\mu_{1,t}}, \ldots, \frac{Q_{N,t}}{\mu_{N,t}} \right) \]

should have a low effective dimension.
### Expected Delays: PCA Analysis

<table>
<thead>
<tr>
<th>Company</th>
<th>% of Variance Explained</th>
<th>% of Variance Explained</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>One Factor</td>
<td>Two Factors</td>
</tr>
<tr>
<td>Alcoa</td>
<td>80%</td>
<td>88%</td>
</tr>
<tr>
<td>American Express</td>
<td>78%</td>
<td>88%</td>
</tr>
<tr>
<td>Boeing</td>
<td>81%</td>
<td>87%</td>
</tr>
<tr>
<td>Bank of America</td>
<td>85%</td>
<td>93%</td>
</tr>
<tr>
<td>Caterpillar</td>
<td>71%</td>
<td>83%</td>
</tr>
<tr>
<td>Cisco</td>
<td>88%</td>
<td>93%</td>
</tr>
<tr>
<td>Chevron</td>
<td>78%</td>
<td>87%</td>
</tr>
<tr>
<td>DuPont</td>
<td>86%</td>
<td>92%</td>
</tr>
<tr>
<td>Disney</td>
<td>87%</td>
<td>91%</td>
</tr>
<tr>
<td>General Electric</td>
<td>87%</td>
<td>94%</td>
</tr>
<tr>
<td>Home Depot</td>
<td>89%</td>
<td>94%</td>
</tr>
<tr>
<td>Hewlett-Packard</td>
<td>87%</td>
<td>92%</td>
</tr>
<tr>
<td>IBM</td>
<td>73%</td>
<td>84%</td>
</tr>
<tr>
<td>Intel</td>
<td>89%</td>
<td>93%</td>
</tr>
<tr>
<td>Johnson &amp; Johnson</td>
<td>87%</td>
<td>91%</td>
</tr>
</tbody>
</table>
State Space Collapse I

Under our model,

\[ ED_{i,t} = \frac{Q_{i,t}}{\mu_{i,t}} = \frac{W_t}{\mu_t} \cdot \frac{1}{\beta_i} \quad \Rightarrow \quad ED_{i,t} = \frac{\beta_{ARCA}}{\beta_i} \cdot ED_{ARCA,t} \]

We test the linear relationship

\[ ED_{i,t} = \alpha_i \cdot \left( \frac{\beta_{ARCA}}{\beta_i} ED_{ARCA,t} \right) + \epsilon_{i,t} \]
State Space Collapse I

Under our model,

\[
ED_{i,t} = \frac{Q_{i,t}}{\mu_{i,t}} = \frac{W_t}{\mu_t} \cdot \frac{1}{\beta_i} \implies ED_{i,t} = \frac{\beta_{ARCA}}{\beta_i} \cdot ED_{ARCA,t}
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ED_{i,t} = \alpha_i \cdot \left( \frac{\beta_{ARCA}}{\beta_i} ED_{ARCA,t} \right) + \epsilon_{i,t}
\]

<table>
<thead>
<tr>
<th>Security</th>
<th>Slope</th>
<th>S.E.</th>
<th>(R^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NASDAQ OMX</td>
<td>0.93</td>
<td>0.0036</td>
<td>0.90</td>
</tr>
<tr>
<td>BATS</td>
<td>0.91</td>
<td>0.0033</td>
<td>0.91</td>
</tr>
<tr>
<td>DirectEdge X (EDGX)</td>
<td>0.97</td>
<td>0.0053</td>
<td>0.82</td>
</tr>
<tr>
<td>NYSE</td>
<td>0.97</td>
<td>0.0055</td>
<td>0.82</td>
</tr>
<tr>
<td>DirectEdge A (EDGA)</td>
<td>0.89</td>
<td>0.0041</td>
<td>0.86</td>
</tr>
</tbody>
</table>
State Space Collapse I

EDGX vs. ARCA $R^2 = 89\%$

BATS vs. ARCA $R^2 = 94\%$

NASDAQ vs. ARCA $R^2 = 95\%$

EDGA vs. ARCA $R^2 = 87\%$

NYSE vs. ARCA $R^2 = 94\%$

(expected delays for WMT, in seconds)
State Space Collapse II

Under our model,

\[ \hat{E}D_t = \frac{W_t}{\mu_t} \cdot \left( \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_N} \right) \]

How much of the variability of ED is explained by \( \hat{E}D \)?

\[ \% \text{ explained} = 1 - \frac{\sum_t ||ED_t - \hat{ED}_t||^2}{\sum_t ||ED_t||^2} \]
State Space Collapse II

Under our model,

$$\hat{\text{ED}}_t = \frac{W_t}{\mu_t} \cdot \left( \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_N} \right)$$

How much of the variability of ED is explained by $\hat{\text{ED}}$?

$$% \text{explained} = 1 - \frac{\sum_t \| \text{ED}_t - \hat{\text{ED}}_t \|^2}{\sum_t \| \text{ED}_t \|^2}$$

<table>
<thead>
<tr>
<th></th>
<th>$R_*^2$</th>
<th></th>
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<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>Alcoa</td>
<td>75%</td>
<td>Home Depot</td>
<td>87%</td>
<td>Merck</td>
<td>78%</td>
</tr>
<tr>
<td>American Express</td>
<td>64%</td>
<td>Hewlett-Packard</td>
<td>77%</td>
<td>Microsoft</td>
<td>80%</td>
</tr>
<tr>
<td>Boeing</td>
<td>75%</td>
<td>IBM</td>
<td>63%</td>
<td>Pfizer</td>
<td>79%</td>
</tr>
<tr>
<td>Bank of America</td>
<td>80%</td>
<td>Intel</td>
<td>82%</td>
<td>Procter &amp; Gamble</td>
<td>80%</td>
</tr>
<tr>
<td>Caterpillar</td>
<td>58%</td>
<td>Johnson &amp; Johnson</td>
<td>83%</td>
<td>AT&amp;T</td>
<td>77%</td>
</tr>
<tr>
<td>Cisco</td>
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<td>JPMorgan</td>
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<tr>
<td>DuPont</td>
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<td>Coca-Cola</td>
<td>81%</td>
<td>Verizon</td>
<td>79%</td>
</tr>
<tr>
<td>Disney</td>
<td>78%</td>
<td>McDonalds</td>
<td>74%</td>
<td>Wal-Mart</td>
<td>85%</td>
</tr>
<tr>
<td>General Electric</td>
<td>82%</td>
<td>3M</td>
<td>62%</td>
<td>Exxon Mobil</td>
<td>81%</td>
</tr>
</tbody>
</table>
We propose a tractable model that incorporates order routing capability and flow heterogeneity with the high-frequency dynamics of LOBs

- characterize equilibrium
- establish state space collapse — fragmented market is coupled through workload
- provide supporting empirical evidence

**Future directions:**

- modeling cancellations
- modeling two-sided markets
- welfare analysis
Dark Pools

Joint work with Ramesh Johari and Kris Iyer.
A Fundamental Tradeoff

In many markets, traders face a choice between: uncertain trade at a better price or guaranteed trade at a worse price.
A Fundamental Tradeoff

In many markets, traders face a choice between: \textit{uncertain trade at a better price} or \textit{guaranteed trade at a worse price}.

We considering a stylized model comparing:

- **Guaranteed market:** (GM)
  - certain trade in an OTC dealer market or electronic limit order book intermediated by dealers or market-makers (e.g., HFT)
  - incurs a transaction cost (“bid-ask spread”)
  - market facilitates price discovery
A Fundamental Tradeoff

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- **Guaranteed market: (GM)**
  certain trade in an OTC dealer market or electronic limit order book intermediated by dealers or market-makers (e.g., HFT) incurs a transaction cost (“bid-ask spread”) market facilitates price discovery

- **Dark pool: (DP)**
  uncertain trade in an electronic crossing network (ECN) trade is dis-intermediated; contra-side investors are crossed zero transaction cost typically refers to external market for prices
“Regulator Probes Dark Pools”

Dark pools have been a source of controversy, especially as they have grown to handle about one in seven stock trades … Most users, as well as regulators, don’t know what is taking place … One area of concern is whether certain dark-pool clients get more information …
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The SEC is also taking a deeper look at dark pools … It is looking at the volume and size of orders that take place in the venues, as well as comparing prices of orders that take place in dark pools with prices on exchanges, among other things …

A rise in off-exchange trading could hurt investors … The reason: With more investors trading in the dark, fewer buy and sell orders are being placed on exchanges. That can translate into worse prices for stocks, because prices for stocks are set on exchanges.
Literature Review

- Extensive literature on information and market microstructure [e.g., Glosten & Milgrom 1985, Kyle 1985, Glosten 1994]

- Theoretical models of dark pools
  [Zhu 2011; Ye 2011; Hendershott & Mendelson 2000]  
  [Dönges & Heinemann 2006; Degryse et al. 2009; Buti et al. 2010]

- Empirical studies on adverse selection & execution guarantees in online markets
  [Anderson et al. 2008; Mathews 2004; Dewan & Hsu, 2004; Lewis 2011]

- Optimal execution in dark pools
  [e.g., Sofianos 2007; Saraiya & Mittal 2009; Kratz & Schöneborn 2010]  
  [Ganchev et al. 2010]
Model

Agents:

• single period, continuum of infinitesimal strategic traders
• valuation = common value + idiosyncratic private value
• can choose to if and where to trade (GM or DP), risk neutral
Model

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Information:
- information (“short-term alpha”) plays an important role
- trader’s own information on short-term price changes
- trader’s impression on informedness of others in market
- fine-grained, heterogeneous signals of future common value
Model

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- single period, continuum of infinitesimal strategic traders
- valuation = common value + idiosyncratic private value
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- information ("short-term alpha") plays an important role
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- trader’s impression on informedness of others in market
- fine-grained, heterogeneous signals of future common value

Equilibrium:

- prices in GM are set by competitive market-makers ("zero-profit")
- Bayes-Nash equilibrium between agents
Results

We compare equilibria with and without a dark pool.
Results

We compare equilibria with and without a dark pool.

Under suitable technical conditions, we establish:

1. **Information segmentation.**
   All else being equal, GM draws more informed traders, while DP draws less informed traders.

2. **Transaction costs.**
   The presence of DP increases the explicit transaction costs of GM.

3. **Adverse selection.**
   The dark pool experiences an implicit transaction cost from the correlation between fill rates and short-term price changes.

4. **Welfare.**
   The presence of DP decreases overall welfare.
Results

We compare equilibria with and without a dark pool.

Under suitable technical conditions, we establish:

(1) **Information segmentation.**
   All else being equal, \textbf{GM} draws more informed traders, while \textbf{DP}
   draws less informed traders.

(2) **Transaction costs.**
   The presence of \textbf{DP} increases the *explicit* transaction costs of \textbf{GM}.
Results

We compare equilibria with and without a dark pool.

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(1) **Information segmentation.**
All else being equal, \textbf{GM} draws more informed traders, while \textbf{DP} draws less informed traders.

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The presence of \textbf{DP} increases the *explicit* transaction costs of \textbf{GM}.

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Results

We compare equilibria with and without a dark pool.

Under suitable technical conditions, we establish:

(1) **Information segmentation.**
All else being equal, $\text{GM}$ draws more informed traders, while $\text{DP}$ draws less informed traders.

(2) **Transaction costs.**
The presence of $\text{DP}$ increases the *explicit* transaction costs of $\text{GM}$.

(3) **Adverse selection.**
The dark pool experiences an *implicit* transaction cost from the correlation between fill rates and short-term price changes.

(4) **Welfare.**
The presence of $\text{DP}$ decreases overall welfare.
The End
Supplementary Slides
The Cost of Latency
Latency Model

\[ \Delta t = \text{Latency} \]

\[ T_0 = 0 \quad \cdots \quad T_i = i\Delta t \quad T_{i+1} \quad T_{i+2} \quad \cdots \quad T = n\Delta t \]
Latency Model

$\Delta t = \text{Latency}$

Suppose a limit order $\ell_i$ is placed at time $T_i = i \Delta t$:

Case 1: Existing limit order $\ell_{i-1}$ gets executed by a market buy ($\ell_{i-1} \leq S T_i + \delta$) in $(T_i, T_i + 1)$.

Case 2: If $S T_i + 1 \geq \ell_i$, then sale occurs at $S T_i + 1$.

Case 3: $\ell_i$ is active over $(T_i + 1, T_i + 2)$. 

$T_0 = 0 \quad \ldots \quad T_i = i \Delta t \quad T_{i+1} \quad T_{i+2} \quad \ldots \quad T = n \Delta t$
Latency Model

\[ \Delta t = \text{Latency} \]

Suppose a limit order \( \ell_i \) is placed at time \( T_i = i\Delta t \):

- **Case 1**: Existing limit order \( \ell_{i-1} \) gets executed by a market buy \((\ell_{i-1} \leq S_{T_i + \delta})\) in \((T_i, T_{i+1})\).

- **Case 2**: If \( S_{T_{i+1}} \geq \ell_i \), then sale occurs at \( S_{T_{i+1}} \).

- **Case 3**: \( \ell_i \) is active over \((T_{i+1}, T_{i+2})\).
Dynamic Programming Decomposition

\[ h_0(\Delta t) \triangleq \sup_{\ell_0, \ell_1, \ldots} E[\text{sale price}] - S_0 \quad (\text{when latency is } \Delta t) \]
Dynamic Programming Decomposition

\[ h_0(\Delta t) \triangleq \sup_{\ell_0, \ell_1, \ldots} E\left[ \text{sale price} \right] - S_0 \quad \text{(when latency is } \Delta t) \]

\[ h_i(\Delta t) \triangleq \sup_{\ell_i, \ell_{i+1}, \ldots} E\left[ \text{sale price} \mid \mathcal{F}_{T_i}, \text{no trade in } [0, T_{i+1}) \right] - S_{T_i} \]
Dynamic Programming Decomposition

\[ h_0(\Delta t) \triangleq \sup_{\ell_0, \ell_1, \ldots} \mathbb{E}[\text{sale price}] - S_0 \]  
(when latency is \( \Delta t \))

\[ h_i(\Delta t) \triangleq \sup_{\ell_i, \ell_{i+1}, \ldots} \mathbb{E}\left[ \text{sale price} \mid \mathcal{F}_{T_i}, \text{no trade in } [0, T_{i+1}] \right] - S_{T_i} \]

Backward recursion:

\[ h_{n-1}(\Delta t) = 0 \]

\[ h_i(\Delta t) = \max_{u_i} F(h_{i+1}(\Delta t), u_i) \]

\[ u_i \triangleq \ell_i - S_{T_i} \]
Intuition

\[ S_0 + \delta \ell' + \delta - C \sigma \sqrt{\Delta t} \ell'' + \delta - \sigma \sum \Delta t \log K \Delta t \]
Intuition

\[ \ell_0 \]

\[ \Delta t \]

\[ \tau_1 \]

\[ T = 2\Delta t \]

\[ S_0 \]

\[ S_0 + \delta \]
Intuition

Execution probability: \( P(\ell_0 \leq S_{\Delta t} + \delta) = \Phi(0) = 1/2 \)
Execution probability: \[ P(\ell_0 \leq S_{\Delta t} + \delta) = \Phi(0) = 1/2 \]
Execution probability: \[ P(\ell'_0 \leq S_{\Delta t} + \delta) = \Phi(C) < 1 \]
Intuition

Execution probability: \( P (\ell_0' \leq S_{\Delta t} + \delta) = \Phi(C) < 1 \)
Intuition

Execution probability: \( P(\ell''_0 \leq S_{\Delta t} + \delta) = \Phi \left( \sqrt{\log K / \Delta t} \right) \rightarrow 1 \)
Lemma.

\[ \ell^*_0 \in \left( S_0 + \delta - \sigma \sqrt{\Delta t \log \frac{K_1}{\Delta t}}, \ S_0 + \delta - \sigma \sqrt{\Delta t \log \frac{K_2}{\Delta t}} \right) \]
Empirical Application: GS

Parameters estimated from Goldman Sachs Group, Inc. (NYSE: GS)

- January 4, 2010
- $S_0 = 170.00$
- $\delta = 0.058$, i.e., 3.4 bp
- $\sigma = 1.92$ (daily), i.e., $\approx 17.9\%$ annualized volatility of returns
- $\mu = 12.03$ (per minute)
- $T = 10$ (seconds)
Empirical Application: GS Optimal Strategy

Limit price premium $u^*_{t_t} = c^*_{t_t} - S_t$ ($\$$)

Time $t$ (sec)

$\Delta t_0 = 0$ (ms)

$\delta$
Empirical Application: GS Optimal Strategy

Limit price premium $u_t^* = \ell_t^* - S_t ($)

- $\Delta t_0 = 0$ (ms)
- $\Delta t_1 = 50$ (ms)

Time $t$ (sec)

$\delta - 2.1\sigma \sqrt{\Delta t_1}$
Empirical Application: GS Optimal Strategy

Limit price premium \( u^*_t = \ell^*_t - S_t (\$) \)

- \( \Delta t_0 = 0 \) (ms)
- \( \Delta t_1 = 50 \) (ms)
- \( \Delta t_2 = 250 \) (ms)

\[ \delta - 2.1 \sigma \sqrt{\Delta t_1} \]
\[ \delta - 1.6 \sigma \sqrt{\Delta t_2} \]
Empirical Application: GS Optimal Strategy

Limit price premium $u^*_t = \ell^*_t - S_t (\$)

- $\Delta t_0 = 0$ (ms)
- $\Delta t_1 = 50$ (ms)
- $\Delta t_2 = 250$ (ms)
- $\Delta t_3 = 500$ (ms)

\[
\begin{align*}
\delta &- 2.1\sigma \sqrt{\Delta t_1} \\
\delta &- 1.6\sigma \sqrt{\Delta t_2} \\
\delta &- 1.4\sigma \sqrt{\Delta t_3}
\end{align*}
\]
Empirical Application: GS Value Function

\[
\begin{align*}
\text{Time } t \text{ (sec)} & \\
\text{Value } h_t(\Delta t) & \\
\Delta t = 0 \text{ (ms)} & \\
\Delta t = 50 \text{ (ms)} & \\
\Delta t = 250 \text{ (ms)} & \\
\Delta t = 500 \text{ (ms)} &
\end{align*}
\]
Empirical Results: NYSE Common Stocks

Parameters from data set of [Aït-Sahalia and Yu, 2009]:

- all NYSE common stocks
- June 1, 1995 to December 31, 2005
- daily estimates of volatility, bid-offer spread

Recent data on Goldman, Sachs

- May 4, 1999 to December 31, 2009
Empirical Results: NYSE Common Stocks

Latency cost (monthly average), $\Delta t = 200$ (ms)

- 90th percentile
- 75th percentile
- Median
- 25th percentile
- GS

Year


0% 5% 10% 15% 20% 25%
Empirical Results: NYSE Common Stocks

Latency cost (monthly average), $\Delta t = 200$ (ms)

- 90th percentile
- 75th percentile
- Median
- 25th percentile
- GS

Year


$1/8$ tick size $1/16$ tick size $0.01$ tick size
Empirical Results: NYSE Common Stocks

Implied latency (ms), LC = 10%

Year

|$1/8$ tick size

|$1/16$ tick size

|$0.01$ tick size

90th percentile

75th percentile

median

25th percentile

GS
Supplementary Slides
Order Routing in Fragmented Markets
Empirical Results

- Dow 30 Stocks; September 2011

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<th>Symbol</th>
<th>Listing Exchange</th>
<th>Price Low (§)</th>
<th>Price High (§)</th>
<th>Average Bid-Ask Spread ($)</th>
<th>Volatility (daily)</th>
<th>Average Daily Volume (shares, (\times 10^6))</th>
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\[ \mu_i = \mu \frac{\beta_i Q_i}{\sum_{i'} \beta_{i'} Q_{i'}} \]

### Attraction Coefficient

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### State Space Collapse II

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The table above shows the linear regressions of the expected delays of each security on a particular exchange, versus that of the benchmark exchange (ARCA) rescaled by the ratio of the attraction coefficients of the two exchanges. This linear relationship is illustrated in particular for the stock of Wal-Mart in Figure 3. Here, we see that the linear relationship holds across all exchanges over periods that vary significantly with respect to their prevailing market conditions, as is manifested in the roughly two orders of magnitude variation in estimated expected delays. While the regression slopes in Figure 3 differ from those predicted by the linear relationship (34), they have the same ordering. That is, the relative slopes of any two exchanges in Figure 3 are inversely ordered according to the corresponding attraction coefficients of the exchanges (cf. Table 4).
Expected Delays vs. Queue Lengths

Delay (norm.)

Queue Length (norm.)
Rate Variability

\[
\frac{\mu}{\Delta} \text{ vs. time of day (minutes)}
\]

- MSFT
- ORCL